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1 Topology

1.1 Definition of a topology

Definition 1.1.1 A topology $T$ on a set $X$ is a subset of $2^X$ which satisfies:
(1) $\emptyset, X \in T$.
(2) If $U_\alpha \in T$ for all $\alpha \in I$, then $\bigcup_{\alpha \in I} U_\alpha \in T$.
(3) If $U_1, \ldots, U_n \in T$, then $\bigcap_{1 \leq i \leq n} U_i \in T$.

$X$ equipped with some topology $T$ is called a topological space. Each $U \in T$ is called an open set (for topology $T$).

Remark 1.1.2 Think about the definition of open sets in $\mathbb{R}^n$.

Definition 1.1.3 For any two different topologies $T$ and $T'$ on $X$, if $T \subset T'$, then we say $T$ is weaker or coarser than $T'$, and $T'$ is finer than $T$.

1.2 Basis (Base) of a topology

Definition 1.2.1 A basis (or base) $\mathcal{B}$ for $X$ is a subset of $2^X$ which satisfies:
(1) For any $x \in X$, there exists $B \in \mathcal{B}$ so that $x \in B$.
(2) For any $x \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}$, there exists $B_3 \in \mathcal{B}$ so that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

Theorem 1.2.2 $T := \{U \subset X : \text{for any } x \in U, \text{ there exists } B \in \mathcal{B} \text{ s.t. } x \in B \subset U\}$ is a topology.

Definition 1.2.3 The topology defined in Theorem 1.2.2 is called the topology generated by basis $\mathcal{B}$.

Remark 1.2.4 Think about the set of all open balls in $\mathbb{R}^n$.

Theorem 1.2.5 The topology $T$ generated by basis $\mathcal{B}$ equals the collection of all unions of elements of $\mathcal{B}$.

Proof. If $U$ belongs to the topology $T$ generated by basis $\mathcal{B}$, then for any $x \in U$, there exists $B_x \in \mathcal{B}$ so that $x \in B_x \subset U$, and we may write $U = \bigcup_{x \in U} B_x$. Conversely, if $U = \bigcup_{\alpha} B_\alpha$, then for each $x \in U$, $x \in B_\alpha \subset U$ for some $\alpha$. \qed
We may think of basis as building blocks of a topology. Sometimes it may not be easy to describe all open sets of a topology, but it is often much easier to find a basis for a topology. Also notice that a topology may be generated by different bases.

**Theorem 1.2.6** Let $\mathcal{B}, \mathcal{B}'$ be bases for $T, T'$, respectively. Then $T \subset T'$ if and only if for any $B \in \mathcal{B}$, $x \in B$, we can find $B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.

**Proof.** (Only if) Pick $B \in \mathcal{B}$, and arbitrary $x \in B$. Since $B \in T \subset T'$, we may find $B' \in \mathcal{B}'$ so that $x \in B' \subset B$.

(If) Let $U \in T$. For any $x \in U$, we may find $B \in \mathcal{B}$ so that $x \in B$, and $B' \in \mathcal{B}'$ s.t. $x \in B' \subset B \subset U$. Therefore, $U = \bigcup_{x \in U} B'_x$, where each $B'_x \in \mathcal{B}'$, which shows $U \in T'$.

---

**1.3 The subspace topology & the product topology on $X \times Y$**

**Definition 1.3.1** Let $Y \subset X$ and $T$ be a topology of $X$. The subspace topology $T_Y$ of $Y$ (induced from $T$) is defined as $T_Y := \{Y \cap U : U \in T\}$.

**Remark 1.3.2** Think about the unit circle with center 0 in $\mathbb{R}^n$.

**Exercise 1.3.3** Find a basis for $T_Y$?

**Definition 1.3.4** Let $T_1, T_2$ be two topologies of $X, Y$, respectively. The product topology $T_1 \times T_2$ on $X \times Y$ is the topology generated by the basis $\mathcal{B} := \{U \times V : U \in T_1, V \in T_2\}$.

**Remark 1.3.5** We may define product topology of $X_1 \times \cdots \times X_n$ similarly. However, things are more complicated when we try to define the product topology on infinitely many (countably or uncountably) product spaces. May see Section 19 of [Munkres].

---

**1.4 Basic topology concepts: limit points, closed sets, and more**

**Definition 1.4.1** A subset $A$ of $X$ is closed if $X \setminus A$ is open.
Exercise 1.4.2 Show that \( \emptyset, X \) are both closed. Show that after taking finitely many union operations or arbitrary intersections of closed sets, we still have closed sets.

Definition 1.4.3 Let \( A \subset X \). The interior of \( A \), denoted by \( \text{int}(A) \), is the union of all open sets contained in \( A \). The closure of \( A \), denoted by \( \overline{A} \), is the intersection of all closed sets containing \( A \).

Theorem 1.4.4 \( A \) is open if and only if \( A = \text{int}(A) \); \( A \) is closed if and only if \( A = \overline{A} \).

Proof. If \( A \) is open, then the union of all open sets contained in \( A \) includes itself. If \( A = \text{int}(A) \), then \( A \) is open by the definition of a topology. The second half is left to the reader. \( \square \)

Definition 1.4.5 Let \( A \) be a subset of the topological space \( X \) with topology \( T \), and \( x \in X \). We say \( x \) is a limit point of \( A \) if for any \( U \in T \) and \( x \in U \), \((U \setminus \{x\}) \cap A \neq \emptyset \).

Remark 1.4.6 Think about the unit circle centered at 0 in \( \mathbb{R}^n \), and the region enclosed by it.

Theorem 1.4.7 Let \( A' \) be the set of all limit points of \( A \). We have \( \overline{A} = A \cup A' \).

Proof. If \( x \in A \), then \( x \in \overline{A} \). If \( x \in A' \), then \( x \notin U \) for any \( U \in T \) s.t. \( A \subset U^c := X \setminus U \) (Otherwise, \((U \setminus \{x\}) \cap A = \emptyset \), a contradiction). This shows \( x \in U^c \) for any \( U \in T \) s.t. \( A \subset U^c \), that is, \( x \in \overline{A} \).

Conversely, If \( x \notin A \cap A' \), then for some \( U \in T \) and \( x \in U \), \( U \cap A = \{x\} \cap A \cup ((U \setminus \{x\}) \cap A) = \emptyset \). Therefore, \( x \notin U^c \), where \( A \subset U^c \), and hence \( x \notin \overline{A} \). \( \square \)

Corollary 1.4.8 A closed set consists of all its limit points.

Proof. The result follows from Theorem 1.4.4 and Theorem 1.4.7. \( \square \)

Theorem 1.4.9 For any subset \( A, B \) of a topological space \( X \), \( \overline{A} \cup B = \overline{A \cup B} \).

Proof. \( \overline{A} \subset \overline{A \cup B} \subset \overline{A} \cup \overline{B} = \overline{A \cup B} \). \( \square \)

Definition 1.4.10 Let \( x, x_1, x_2, \ldots, x_n, \ldots \in X \). We say \( x_n \) converges to \( x \), denoted by \( x_n \to x \), if for every \( U \in T \) s.t. \( x \in U \), there exists \( N \in \mathbb{N} \) so that for all \( n > N \), we have \( x_n \in U \).

Exercise 1.4.11 Show that if \( x_n \to x \), where \( x_n \in A \) and \( x_n \neq x \) for all \( n \in \mathbb{N} \), then \( x \) is a limit point of \( A \). Conversely, If \( x \) is a limit point of \( A \), then there exists a sequence
\{x_n\} \subset A \text{ so that } x_n \to x.

**Example 1.4.12 (Multiple limits)** Let \( X = \{1, 2, 3\}, \ T = \emptyset, X, \{1, 2\}, \{2, 3\}, \{2\} \). What is the limit of the sequence \( \{2, 2, \cdots, 2, \cdots\} \)?

We now introduce a new topology concept to make sure we always have unique limit points.

**Definition 1.4.13 (Hausdorff space)** A topological space \( X \) with topology \( T \) is called a Hausdorff space if for any \( x \neq y, \ x, y \in X \), we can find \( U, V \in T \) so that \( x \in U, \ y \in V \), and \( U \cap V = \emptyset \).

**Theorem 1.4.14** If \( X \) is a Hausdorff space, then any sequence of points of \( X \) converges to at most one point in \( X \).

**Proof.** Straightforward. (What happens if the sequence converges to two distinct points?)

**Lemma 1.4.15** Let \( X \) be Hausdorff. Then for every \( x \in X \), \( \{x\} \) is a closed set.

**Proof.** Try to write \( X \setminus \{x\} \) as a union of open sets.

**Theorem 1.4.16** Let \( X \) be a Hausdorff space, and \( A \subset X \). Then \( x \in X \) is a limit point of \( A \) if and only if every neighborhood of \( x \) contains infinitely many points of \( A \). (Neighborhood of \( x \): open set containing \( x \))

**Proof.** If part is trivial. For only if part, assume that for some neighborhood \( U \) of \( x \), there’re only finitely many points of \( A \) in \( U \), say, \( x_1, \cdots, x_N \). By Lemma 1.4.15, \( U \setminus \{x_1, \cdots, x_N\} \) is an open set. We have \( ((U \setminus \{x_1, \cdots, x_N\}) \setminus \{x\}) \cap A = \emptyset \), which shows that \( x \) is not a limit point of \( A \).

**Remark 1.4.17** The assumptions made in Theorem 1.4.16 can be weakened (It suffices to assume that \( X \) is a \( T_1 \) space, see Definition 1.9.1). See also [Munkres].

**Theorem 1.4.18** Let \( T \) and \( T' \) are two distinct topologies for \( X \) s.t. \( T \subset T' \). If \( x_n \to x \) under \( T' \), then \( x_n \to x \) under \( T \).

**Remark 1.4.19** Theorem 1.4.18 shows it is easier for a sequence to converge in a weaker topology. We would say the convergence is weaker in weaker topology.
1.5 Metric topology

Definition 1.5.1 A metric on a set $X$ is a function $d : X \times X \to \mathbb{R}$ so that
\begin{enumerate}
\item $d(x, y) > 0$ for all $x \neq y$, $d(x, x) = 0$.
\item $d(x, y) = d(y, x)$.
\item $d(x, y) + d(y, z) \geq d(x, z)$.
\end{enumerate}

Definition 1.5.2 A topological space $X$ with topology $T$ is called a metric space if $T$ is generated by the collection of balls (which forms a basis) $B(x, \epsilon) := \{y : d(x, y) < \epsilon\}$, $x \in X$, $\epsilon > 0$. Whenever there is a metric $d$ s.t. $T$ is generated this way, we say $X$ is metrizable.

Definition 1.5.3 Let $(X, d)$ be a metric space. We say $A \subset X$ is said to be bounded if there exists $M > 0$ s.t. $d(a, b) < M$ for all $a, b \in A$.

Definition 1.5.4 A metric space $(X, d)$ is said to be totally bounded if there exists finitely many balls of the same radius that covers $X$.

Definition 1.5.5 Let $(X, d)$ be a metric space. A Cauchy sequence is a sequence $\{x_n\} \subset X$ s.t. for every $\epsilon > 0$, there exists $N \in \mathbb{N}$, we have $d(x_n, x_m) < \epsilon$ for all $n, m > N$. A metric space $X$ is called a complete metric space if every Cauchy sequence in $X$ converges to some point in $X$.

1.6 Continuous functions

Definition 1.6.1 Let $X, Y$ be topological spaces. A function $f : X \to Y$ is said to be continuous if for any $U$ open in $Y$, $f^{-1}(U)$ is open in $X$.

Theorem 1.6.2 Let $X, Y$ be topological spaces, and $f : X \to Y$, then TFAE:
\begin{enumerate}
\item $f$ is continuous.
\item For any $U$ closed in $Y$, $f^{-1}(U)$ is closed in $X$.
\item For every $x \in X$ and each neighborhood $V$ of $f(x)$, there exists a neighborhood $U$ of $x$ s.t. $f(U) \subset V$.
\end{enumerate}

If, furthermore, both $X$ and $Y$ are metric spaces, then all the three statements are equivalent to the following one:

\begin{enumerate}
\item For every $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ s.t. $f(B(x, \delta)) \subset B(f(x), \epsilon)$.
\end{enumerate}

Proof. We’ll prove the equivalence of (1), (2), and (3), and the rest are left as exercises. First of all, (1) $\iff$ (2) is straightforward. For (1) $\Rightarrow$ (3), we may simply let $U = f^{-1}(V)$, and we can
easily find \( x \in U \) and \( U \) is open. For (3) \( \Rightarrow \) (1), for any \( V \) open in \( Y \), and \( x \in X \), we may find a neighborhood \( U_x \) for \( x \) s.t. \( f(U_x) \subset V \), and this implies \( x \in U_x \subset f^{-1}(f(U_x)) \subset f^{-1}(V) \). Therefore, \( f^{-1}(V) \) equals the union of all these \( U_x \)'s, and thus \( f^{-1}(V) \) is open in \( X \).

**Definition 1.6.3** Let \( X, Y \) be topological spaces. A function \( f : X \to Y \) is called a homeomorphism if both \( f \) and \( f^{-1} \) are continuous. If such homeomorphism exists between \( X \) and \( Y \), then we say \( X \) is homeomorphic to \( Y \).

**Remark 1.6.4** All topological properties are preserved by homeomorphisms. We may see homeomorphism as an equivalence relation between topological spaces.

### 1.7 Compactness

**Definition 1.7.1** A subset \( A \) of a topological space \( X \) is said to be compact if every open covering of \( A \) contains a finite subcover.

**Theorem 1.7.2** Every closed subset \( C \) of a compact set \( D \) is compact.

*Proof.* Let \( \mathcal{A} \) be an open covering of \( C \). We have that \( \{ \mathcal{A}, C^c \} \) is an open covering of \( D \). By compactness of \( D \), we can find a finite subcover \( \{ U_1, \ldots, U_n, C^c \} \) that covers \( D \). It turns out that \( \{ U_1, \ldots, U_n \} \) covers \( C \).

**Theorem 1.7.3** Any finite union of compact sets is compact.

*Proof.* It suffices to show that if \( C \) and \( D \) are compacts, then so is \( C \cup D \). Let \( \mathcal{A} = \bigcup_{\alpha \in I} U_\alpha \) be an open covering of \( C \cup D \). Therefore, there exists \( \bigcup_{j=1}^N U_{n_j} \) and \( \bigcup_{j=1}^M U_{m_j} \) that covers \( C \), \( D \), respectively. \( \bigcup_{j=1}^N U_{n_j} \cup \bigcup_{j=1}^M U_{m_j} \) is therefore a finite subcover of \( C \cup D \).

**Theorem 1.7.4** In a Hausdorff space \( X \), every compact subset \( D \) is closed.

*Proof.* Fix \( x \in X \setminus D \). For any \( y \in D \), we may find some neighborhood \( U_y \) of \( y \) and \( V_{y,x} \) of \( x \) so that \( U_y \cap V_{y,x} = \emptyset \) (Since \( X \) is Hausdorff). Since \( D \) is compact, we may find finitely many such \( U_y \)'s that cover \( D \), Say \( U_{y_1}, \ldots, U_{y_N} \). Since \( U_{y_1} \cup \cdots \cup U_{y_N} \) and \( V_{y_1,x} \cap \cdots \cap V_{y_N,x} \) are disjoint, it follows that \( V_{y_1,x} \cap \cdots \cap V_{y_N,x} \) is a neighborhood of \( x \) which lies in \( X \setminus D \). This proves that \( X \setminus D \) is open, and thus \( D \) is closed.

**Remark 1.7.5** Theorem 1.7.4 is false if the underlying space is not Hausdorff. Again, consider \( X = \{1, 2, 3\} \), \( T = \{\emptyset, X, \{1, 2\}, \{2, 3\}, \{2\}\} \). Every subset of \( X \) is a compact set, but not every subset of \( X \) is a closed set.

**Corollary 1.7.6** In a Hausdorff space \( X \), arbitrary intersection of compact sets is still compact.
Proof. By theorem 1.7.4, arbitrary intersection of compact sets is closed. By Theorem 1.7.2, it is compact.

**Theorem 1.7.7** Continuous functions map compact sets to compact sets.

Proof. Let $f : X \to Y$ be a continuous function, $X, Y$ are topological spaces. Let $A$ be a compact set in $X$. For any open covering $\{U_\alpha\}$ of $f(A)$, $\{f^{-1}(U_\alpha)\}$ is an open covering of $f^{-1}(f(A))$ and hence of $A$. Compactness of $A$ implies that $A \subset \bigcup_{j=1}^n f^{-1}(U_j)$. Therefore, $f(A) \subset f(\bigcup_{j=1}^n f^{-1}(U_j)) = \bigcup_{j=1}^n f(f^{-1}(U_j)) = \bigcup_{j=1}^n U_j$, which shows $f(A)$ has a finite subcover of $\{U_\alpha\}$.

**Exercise 1.7.8** Show that $D$ is compact if and only if for any collection $C$ of closed subsets of $D$ s.t. the intersection of any finite subcollection of $C$ is nonempty, the intersection of all elements in $C$ is nonempty.

**Definition 1.7.9** A subset $D$ of a topological space $X$ is said to be **limit point compact** if every infinite subset $A$ of $D$, there exists a point $x \in D$ so that $x$ is a limit point of $A$.

**Definition 1.7.10** A subset $D$ of a topological space $X$ is said to be **sequentially compact** if every sequence of $D$ contains a convergent subsequence which converges to some point $x$ in $D$ ($x$ does not necessarily belong to the sequence).

**Theorem 1.7.11** If a subset $D$ of a topological space $X$ is sequentially compact, then it is limit point compact.

Proof. Let $A$ be an infinite subset of $D$. We may pick countably many of them, and list them as a subsequence $\{x_n\}$. By the assumption we have $x_n \to x \in D$, so every neighborhood $U_x$ of $x$ must contain a point other than $x$. This shows $(U_x \setminus \{x\}) \cap A \neq \emptyset$.

**Theorem 1.7.12** If a subset $D$ of a topological space $X$ is compact, then it is limit point compact.

Proof. Let $A$ be an infinite subset of $D$. If there is no $x \in D$ s.t. $x$ is a limit point of $A$, then for each $x \in A \subset D$, we may find some neighborhood $U_x$ of $x$ so that $U_x \cap A = \{x\}$. All these $U_x$’s form an open covering of $D$, but it is easily seen that there is no finite subcover of this open covering. Therefore, there must be some $x \in D$ s.t. $x$ is a limit point of $A$, and this proves $D$ is limit point compact.

**Remark 1.7.13** In general topological spaces, we do have examples that neither of the converse propositions of Theorem 1.7.8 and Theorem 1.7.9 is true.

**Theorem 1.7.14** The following statements are equivalent for any subset $D$ of the metric space $(X, d)$:
(1) $D$ is compact.
(2) $D$ is complete and totally bounded.
(3) $D$ is limit point compact.
(4) $D$ is sequentially compact.

Proof. For (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), see [Dudley]. For (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (1), see [Munkres].

Remark 1.7.15 Recall Heine-Borel theorem: In $\mathbb{R}^n$, a set is a compact set if and only if it is closed and bounded.

### 1.8 Countability axioms

**Definition 1.8.1** A topological space $X$ is called **first countable** if $X$ has a countable basis at each $x \in X$, which means there exists a basis $\mathcal{B}$ that generates the topology, and there exists a countable subcollection of $\mathcal{B}$ so that $x$ falls in each of them, and every neighborhood of $x$ contains at least one basis element of this countable subcollection.

**Definition 1.8.2** A topological space $X$ with topology $T$ is called **second countable** if there exists a countable basis that generates $T$.

**Theorem 1.8.3** Second countability of a topological space implies first countability

**Definition 1.8.4** A subset $A$ is **dense** in a topological space $X$ if $\overline{A} = X$.

**Definition 1.8.5** A topological space $X$ is said to be **separable** if there exists $A$ s.t. $\overline{A} = X$, and $A$ has only countably many elements.

**Theorem 1.8.6** A subset $A$ is dense in a topological space $X$ if and only if for every $x \in X$ and every open set $U_x$ containing $x$, $U_x$ contains at least one element in $A$.

Proof. By Theorem 1.4.7, we have $\overline{A} = A \cup A'$. If $A$ is dense in $X$, then every $x$ is either in $A$ or is a limit point of $A$, and in either case it is easily seen that the second assertion of the theorem is true. Conversely, for any $x \in X$, if every neighborhood $U_x$ of $x$ satisfies $(U_x \setminus \{x\}) \cap A = \emptyset$, then $x \in A'$. If there is some neighborhood $U_x$ of $x$ s.t. $(U_x \setminus \{x\}) \cap A = \emptyset$, then $x \in A$ (Otherwise we would have $U_x \cap A = \emptyset$, which contradicts our assumption). Therefore, $x \in A \cup A' = \overline{A}$. □

**Theorem 1.8.7** If a topological space $X$ is second countable, then it is separable.

Proof. Let $\mathcal{B} = \{B_1, B_2, \cdots\}$ be a countable basis for $X$. For each $B_j$, we may pick some $x_j \in B_j$. The set $\{x_1, x_2, \cdots\}$ is dense in $X$. □
Theorem 1.8.8 Any metric space \((X, d)\) is first countable.

Proof. For each \(x \in X\), consider \(B(x, \epsilon) = \{y : d(x, y) < \epsilon\}\), where \(\epsilon \in \mathbb{Q}^+\).

Theorem 1.8.9 Any separable metric space \((X, d)\) is second countable.

Proof. Let \(A\) be a countable dense subset of \(X\). Consider the collection of \(B(x, \epsilon) = \{y : d(x, y) < \epsilon\}\), where \(\epsilon \in \mathbb{Q}^+, x \in A\). The goal is to show that the collection of these balls form a basis, which is left as an exercise.

Corollary 1.8.10 Any subset \(A\) of a separable metric space \(X\) is also separable with the subspace topology induced from \(X\).

Proof. By the previous theorem, \(X\) has a countable basis. The subspace topology on \(A\) is thus generated by a countable basis, too. By Theorem 1.8.7, \(A\) is separable.

Theorem 1.8.11 Any compact metric space \((X, d)\) is second countable.

Proof. For every \(n \in \mathbb{N}\), we may cover \(X\) with finitely many balls \(B(x, 1/n)\), say \(B(x_n, 1/n)\). The collection of all these balls form a countable basis. (Details are left to the readers.)

The next theorem is an application of first countability.

Theorem 1.8.12 Let \(X, Y\) be topological spaces. If the function \(f\) is continuous, then for any convergent sequence \(x_n \to x\) in \(X\), the sequence \(f(x_n) \to f(x)\) in \(Y\). The converse is true if \(X\) is first countable.

Proof. Let \(f\) be continuous. By Theorem 1.6.2, for all \(x \in X\), and each neighborhood \(V\) of \(f(x)\), there exists a neighborhood \(U\) of \(x\) s.t. \(f(U) \subset V\). Now, for any sequence \(\{x_n\}\) s.t. \(x_n \to x, x_n \in U\) for all \(n\) large, and hence \(f(x_n) \in V\) for all \(n\) large. This implies \(f(x_n) \to f(x)\).

Let \(C\) be a closed set in \(Y\) and \(x\) be a limit point of \(f^{-1}(C)\). Since \(X\) is first countable, we may find a countable collection of basis elements \(\{B'_1, B'_2, \ldots\}\) so that for any open set \(U\) containing \(x\), \(U\) contains at least one of these \(B'_i\)'s. Now we let \(B_1 = B'_1, B_2 = B'_1 \cap B'_2, B_3 = B'_1 \cap B'_2 \cap B'_3\), and so forth. Since \(x\) is a limit point of \(f^{-1}(C)\), we may pick some \(x_j \in (B_j \setminus \{x\}) \cap C\) for every \(j \in \mathbb{N}\). It’s not hard to see from definition that \(x_n \to x\). Therefore, \(f(x_n) \to f(x)\) (Note that \(f(x_n) \in C\) for all \(n \in \mathbb{N}\)), and this shows \(f(x) \in \overline{C} = C\) (By Theorem 1.4.7). Therefore, \(x \in f^{-1}(C)\), and hence \(f^{-1}(C)\) is closed.
1.9 Separation axioms

**Definitions 1.9.1** Let \( X \) be a topological space.

1. \( X \) is said to be a \( T_1 \) space if all singletons \( \{x\} \) are closed.
2. \( X \) is said to be a \( T_2 \) space if it is a Hausdorff space (See Definition 1.4.12).
3. A \( T_1 \) space \( X \) is said to be a \( T_3 \) space, or a regular space, if for any \( x \in X \), \( C \) closed in \( X \), \( x \notin C \), we can find open sets \( U, V \) so that \( x \in U \), \( C \subset V \), and \( U \cap V = \emptyset \).
4. A \( T_1 \) space \( X \) is said to be a \( T_4 \) space, or a normal space, if for any \( C_1, C_2 \) closed in \( X \), \( C_1 \cap C_2 = \emptyset \), we can find open sets \( U, V \) so that \( C_1 \in U \), \( C_2 \subset V \), and \( U \cap V = \emptyset \).

**Remark 1.9.2** There're also \( T_{3\frac{1}{2}} \) (completely regular) and \( T_5 \) (completely normal) topologies, which are not discussed here. Also, it is obvious that \( T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \) (See also Lemma 1.4.14).

**Theorem 1.9.3** Every regular space \( X \) with a countable basis (2nd countable) is normal.

*Proof.* See [Munkres].

**Theorem 1.9.4** Every metric space \((X, d)\) is normal.

*Proof.* Let \( C_1 \) and \( C_2 \) be two closed sets in \( X \). For each \( x \in C_1 \), Pick a tiny ball \( B(x, \epsilon_x) \) so that \( B(x, \epsilon_x) \cap C_2 = \emptyset \). Similarly, for each \( y \in C_2 \), Pick a tiny ball \( B(y, \epsilon_y) \) so that \( B(y, \epsilon_y) \cap C_1 = \emptyset \). We claim that \( U_1 := \bigcup_{x \in C_1} B(x, \epsilon_x/3) \) and \( U_2 := \bigcup_{y \in C_2} B(y, \epsilon_y/3) \) are two open sets that separates \( C_1 \) and \( C_2 \). The only nontrivial thing we need to show is that \( U_1 \cap U_2 = \emptyset \).

If \( z \in U_1 \cap U_2 \), then \( z \in B(x, \epsilon_x/3) \) and \( y \in B(y, \epsilon_y/3) \) for some \( x \in C_1 \) and \( y \in C_2 \). From triangle inequality we have \( d(x, y) < \epsilon_x/3 + \epsilon_y/3 \leq \max(2\epsilon_x/3, 2\epsilon_y/3) \). WLOG we assume that \( \epsilon_x = \max(\epsilon_x, \epsilon_y) \), and hence we have \( B(x, \epsilon_x) \cap C_2 \neq \emptyset \), but this is a contradiction to our assumption.

**Theorem 1.9.5** Every compact Hausdorff space \( X \) is normal.

*Proof.* Let \( C_1 \) and \( C_2 \) be two closed sets in \( X \). As closed subsets of a compact set, \( C_1 \) and \( C_2 \) are both compact (Theorem 1.7.2). For each \( x \in C_1 \), we may find two open sets \( V_x \) containing \( C_2 \) and \( U_x \) containing \( x \) so that \( U_x \cap V_x = \emptyset \) (Why?). Cover \( C_1 \) with finitely many \( U_{x_1} \cdots U_{x_N} \), we find that \( U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_N} \) and \( V_{x_1} \cap V_{x_2} \cap \cdots \cap V_{x_N} \) are two open sets that separates \( C_1 \) and \( C_2 \).

**Theorem 1.9.6** (Urysohn metrization theorem) Every second countable regular space \( X \) is metrizable.

*Proof.* See [Munkres].
Now we state two famous theorems, the Stone-Weierstrass theorem and the Urysohn’s lemma, before we proceed to Theorem 1.9.9.

**Lemma 1.9.7 (Stone-Weierstrass Theorem)** Let $X$ be a compact Hausdorff space, $C(X, \mathbb{R})$ be the set of all real-valued continuous functions on $X$. Let $\mathcal{A}$ be a collection of functions which satisfies

1. For any $a, b \in \mathbb{R}$, $f, g \in \mathcal{A}$, $af + bg \in \mathcal{A}$.
2. For any $f, g \in \mathcal{A}$, $fg \in \mathcal{A}$.
3. For any $x, y \in X$, we may find some $f \in \mathcal{A}$ so that $f(x) \neq f(y)$.
4. $\mathcal{A}$ contains all constant functions.

Then $\mathcal{A}$ is dense in $C(X, \mathbb{R})$ with supremum norm.

*Proof.* See [Peter Lax], p126.

**Lemma 1.9.8 (Urysohn’s lemma)** Let $X$ be a normal space; let $A$ and $B$ be disjoint closed subsets of $X$. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map $f : X \to [a, b]$ such that $f(x) = a$ for every $x \in A$, and $f(x) = b$ for every $x \in B$.

*Proof.* See [Munkres].

**Theorem 1.9.9** For any compact Hausdorff space $X$, the following are equivalent:

1. $X$ is second countable.
2. $X$ is metrizable.
3. The space $C(X, \mathbb{R})$ with the supremum norm is separable.

*Proof.* (1) $\Rightarrow$ (2): By both Theorem 1.9.5 and Theorem 1.9.6. (2) $\Rightarrow$ (1): By Theorem 1.8.11.

For (2) $\Rightarrow$ (3): From the proof in Theorem 1.8.11, we know that a basis that generates the topology of $X$ is given by a countable collection of balls $B(x, 1/n)$, $n \in \mathbb{N}$, and we list them as $\mathcal{B} := \{B_1, B_2, \cdots \}$. For each $i \neq j \in \mathbb{N}$, when $\overline{B_i} \subseteq B_j$, we apply Lemma 1.9.8 to define $f_{i,j}(x) = f_{j,i}(x) = 1$ if $x \in \overline{B_i}$, $f_{i,j}(x) = f_{j,i}(x) = 0$ if $x \notin B_j$, and $f_{i,j} \equiv f_{j,i} \in C(X, \mathbb{R})$. For all other cases of $B_i, B_j$ we simply define $f_{i,j} \equiv 1$. From the definition of $\mathcal{A}$ we know that all constant functions lie in $\mathcal{A}$. Therefore, by Theorem 1.9.7, $\mathcal{A}$ is dense in $C(X, \mathbb{R})$ with supremum norm.

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Let $\mathcal{A}'$ be the collection of all possible linear combinations (with rational coefficients) of products of finitely many functions $f_{i,j}$. It’s not hard to see $\mathcal{A}'$ is dense in $\mathcal{A}$ with sup norm, and thus $\mathcal{A}'$ is dense in $C(X, \mathbb{R})$ with sup norm. This proves (3).

For $(3) \Rightarrow (1)$: Let $\{f_n : n \in \mathbb{N}\}$ be dense in $C(X, \mathbb{R})$ (with sup norm). We claim that $\mathcal{B}$ defined to be the collection of all finite intersections of sets of form $f_n^{-1}(p, q)$, $n \in \mathbb{N}$, $p, q \in \mathbb{Q}$, is a basis.

To see this, for any open set $U \in X$ and $x \in U$, define $f(x) = 1$ and $f(y) = 0$ for all $y \not\in U$, so that $f \in C(X, \mathbb{R})$ by Theorem 1.9.8 (Urysohn’s lemma). We may find some $N$ s.t.

$$x \in f_N^{-1}(0.9, 1.1) \subset f^{-1}(0.8, 1.2) \subset U.$$

Besides, $\mathcal{B}$ is closed under finite intersection by its definition. Thus $\mathcal{B}$ is a countable basis, and (1) is proved. \qed