Some results for locally compact Hausdorff spaces

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The following materials are all taken from [Rudin]. Other references: [Munkres]; [Folland] sec 4.5; [Richard Bass] sec 20.9

1 General Stuff

Theorem 1.1. Suppose $U$ is open in a locally compact Hausdorff space $X$, $K \subset U$, and $K$ is compact. Then there is an open set $V$ with compact closure s.t. $K \subset V \subset \overline{V} \subset U$. [Rudin p.37]

Theorem 1.2. (Urysohn’s lemma) Suppose $X$ is a locally compact Hausdorff space, $V$ is open in $X$, $K \subset V$, and $K$ is compact. Then there exists an $f \in C_c(X)$ s.t. $\chi_K \leq f \leq \chi_V$. [Rudin p.39]

Remark 1.3. See also [Munkres] for another version of Urysohn’s lemma, which assumes that $X$ is normal. The arguments are also slightly different.

Theorem 1.4. (Partition of unity) Suppose $V_1, \ldots, V_n$ are open subsets of a locally compact Hausdorff space $X$, $K$ is compact, and $K \subset V_1 \cup \cdots \cup V_n$. Then there exists $0 \leq h_i < \chi_{V_i}$, $1 \leq i \leq n$, so that $h_1(x) + \cdots + h_n(x) = 1$ for all $x \in K$. [Rudin p.40]

Remarks 1.5. (1) Must take a look at the original proof to see if $h_i$ is continuous and compact supported. See also [245B, Notes 12: Continuous functions on locally compact Hausdorff spaces] of Terence Tao’s website. (2) Theorem 1. $\Rightarrow$ Theorem 2. $\Rightarrow$ Theorem 4. (3) See [Munkres] page 225. for
another version of partition of unity (on normal spaces).

Remarks 1.6. I skip the Riesz representation theorem.

Theorem 1.7. (Lusin’s theorem) Suppose $X$ is a locally compact Hausdorff space, $\mu$ is a Radon measure on $X$, and $\mu$ is complete. If $f$ is a complex measurable function on $X$, and there exists $A$ s.t. $\mu(A) < \infty$, $f(x) = 0$ for $x \notin A$, then, given $\epsilon > 0$, there exists some $g \in C_c(X)$ so that $\mu(g \neq f) < \epsilon$, and we may arrange it so that $\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|$.

[Rudin p.55]

Remark 1.8. Wikipedia says Lusin’s theorem holds for second countable topological space. (?)

Theorem 1.9. Suppose $X$ is a locally compact Hausdorff space, $\mu$ is a Radon measure on $X$, and $\mu$ is complete. For $1 \leq p < \infty$, $C_c(X)$ is dense in $L^p(\mu)$. [Rudin p.69]

Definition 1.10. Let $X$ be a locally compact Hausdorff space. A complex function $f$ is said to vanish at infinity if for every $\epsilon > 0$, there exists a compact set $K \subset X$ s.t. $|f(x)| < \epsilon$ for all $x \in X \setminus K$. The class of all continuous function $f$ on $X$ which vanish at infinity is called $C_0(X)$. [Rudin p.70]

Theorem 1.11. Suppose $X$ is a locally compact Hausdorff space. $C_0(X)$ is the completion of $C_c(X)$ w.r.t. supremum norm. [Rudin p.70]

2 One-point compactification

Theorem 2.1. Let $(X,T)$ be a Hausdorff space. Let $\infty$ be a point not in $X$, and let $X^* := X \cup \{\infty\}$. Define $T^* := T \cup T'$, where $T' := \{U \subset X^* : X^* \setminus U$ compact in $(X,T)\}$. We claim that $T^*$ is a topology for $X^*$.

Proof. (1) $\varnothing \in T \subset T^*$. Since $X^* \setminus X^* = \varnothing$, which is compact in $(X,T)$, $X^* \in T' \subset T^*$.

(2) Let $\{U_\alpha\}_{\alpha \in I} \subset T'$. We may write each $U_\alpha = (X \setminus K_\alpha) \cup \{\infty\}$, $K_\alpha$ is compact in $(X,T)$. We have $\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} (X \setminus K_\alpha) \cup \{\infty\} = (X \setminus \bigcap_{\alpha \in I} K_\alpha) \cup \{\infty\}$, which is still in $T'$ (The intersection of compacts is a compact set since $X$ is Hausdorff). For $A_1 \in T$, $A_2 = (X \setminus K) \cup \{\infty\} \in T'$,
$A_1 \cup A_2 = X \setminus ((X \setminus A_1) \cap K) \in T' \subset T^*$, where $(X \setminus A_1) \cap K$ is compact again because $X$ is Hausdorff.

(3) Let $U_1, \cdots , U_n \in T'$, and we write each $U_j = (X \setminus K_j) \cup \{\infty\}$, $K_j$ is compact in $(X,T)$. Therefore, $\bigcap_{j=1}^{n} U_j = \bigcap_{j=1}^{n} (X \setminus K_j) \cup \{\infty\} = (X \setminus \bigcup_{j=1}^{n} K_j) \cup \{\infty\} \in T' \subset T^*$. For $A_1 \subseteq T$, $A_2 = (X \setminus K) \cup \{\infty\} \in T'$, $A_1 \cap A_2 = (X \setminus K) \cap A_1 \in T \subseteq T^*$. \hfill \Box

When $(X,T)$ is a locally compact Hausdorff space, we have the following result:

**Theorem 2.2.** Let $(X,T)$ be a locally compact Hausdorff space. Then the one-point compactification $(X^*,T^*)$ as shown in Theorem 2.1 is a compact Hausdorff space. [Richard Bass, real analysis, Sec 20.9]

**Remark 2.3.** $(X,T)$ coincides with the subspace topology induced from $(X^*,T^*)$.

**Theorem 2.4.** Let $(X,T)$ be a locally compact Hausdorff space. Then every function $f : E \rightarrow \mathbb{R}$ belongs to $C_0(X)$ if and only if the extension $\hat{f} : X^* \rightarrow \mathbb{R}$, $\hat{f}(\infty) = 0$, $\hat{f}(x) = f(x)$ for $x \in X$ is a continuous function on $(X^*,T^*)$.

**Proof.** Let $f \in C_0(X)$. For any $V$ open in $\mathbb{R}$, if $0 \notin V$, then $\hat{f}^{-1}(V) = f^{-1}(V) \in T$. If $0 \in V$, then $\hat{f}^{-1}(V) \setminus \{\infty\} = f^{-1}(V) \supset X \setminus K$ for some compact set $K$ in $(X,T)$. It follows that $X \setminus f^{-1}(V)$ is a compact set in $(X,T)$ (since it is a closed subset of $K$). As a result, $\hat{f}^{-1}(V) = \{\infty\} \cup f^{-1}(V) = \{\infty\} \cup (X \setminus f^{-1}(V))) \in T' \subset T^*$, which proves $\hat{f} \in C(X^*)$.

Conversely, let $\hat{f} \in C(X^*)$. For any $V$ open in $\mathbb{R}$, if $0 \notin V$, then $f^{-1}(V) = \hat{f}^{-1}(V) \in T$. If $0 \in V$, then $\hat{f}^{-1}(V) = (X \setminus K) \cup \{\infty\}$, $K$ compact in $(X,T)$, and thus $f^{-1}(V) = X \setminus K \in T$. This proves $f$ is continuous on $(X,T)$, and actually, that $f^{-1}(V) = X \setminus K$ for $0 \in V$ is exactly equivalent to the fact that $f$ vanishes at infinity. \hfill \Box

**Theorem 2.5.** Let $(X,T)$ be a locally compact Hausdorff second countable space (LCCB). Then the one-point compactification $(X^*,T^*)$ as shown in Theorem 2.1 is second countable.

**Proof.** We first notice that the collection $\mathcal{C}$ of all open sets with compact closure form a basis for $T$. To see $\mathcal{C}$ is a basis, first, from the definition of locally compact space we know that every $x \in X$ is covered by some open set
in $C$. Second, for $C_1, C_2 \in C$, and $x \in C_1 \cap C_2$, there exists some open neighborhood of $U$ of $x$ so that $U \subset C_1 \cup C_2$, and we have $U \subset \overline{C_1 \cup C_2} = \overline{C_1} \cup \overline{C_2}$. This implies $U$ is compact, and $U \in C$. To see $C$ generates $T$, for any open set $U$ that contains $x$, there exists some $V$ with compact closure s.t. $x \in V \subset U \subset V \subset U$ (Theorem 1.1).

Since $(X, T)$ is second countable, every basis has a countable subfamily that is still a basis for $T$. Therefore, we may apply this result to $C$ above to get a countable basis $B := \{B_1, B_2, \cdots\}$, where each $B_j \in B$ has compact closure.

Define $B^* := B \cup B'$, where $B' := \{(X \setminus \bigcup_{j=1}^N \overline{B_{n_j}}) \cup \{\infty\} : \{B_{n_j}\}_j$ is a subsequence of $\{B_n\}_n, N \in \mathbb{N}\}$. To see $B^*$ is a basis in $X^*$, first we note that for every $x \in X^*$ is covered by some $B \in B^*$. Second, for $B_1, B_2 \in B$ (resp. $B'$), $x \in B_1 \cap B_2$, there exists some $B_3 \in B$ (resp. $B'$) so that $x \in B_3 \subset B_1 \cap B_2$. For $B_1 \in B, B_2 \in B'$, $B_1 \cap B_2$ is an open set in $X$, so there exists some $B_3 \in B$ so that $x \in B_3 \subset B_1 \cap B_2$ (Since $B$ is a basis for $(X, T)$). To see $B^*$ generates $T^*$, the only nontrivial fact to prove is that for any open set $U$ in $T^*$ of form $(X \setminus K) \cup \{\infty\}$, $K$ compact in $(X, T)$, $x = \{\infty\} \in U$, there exists some $B \in B'$ so that $x \in B \subset U$. This is due to the fact that since $K$ is compact in $(X, T)$, we may find some $\{B_{m_1}, \cdots, B_{m_k}\} \subset B$ that covers $K$, and it follows that $x \in (X \setminus \bigcup_{j=1}^k \overline{B_{m_j}}) \cup \{\infty\} \subset U$.

Therefore, $B^*$ is a countable basis for $(X^*, T^*)$, and the proof is complete.

Corollary 2.6. Let $(X, T)$ be a locally compact Hausdorff second countable space (LCCB). Then $X$ is metrizable.

Proof. By the previous theorems, the one-point compactification $(X^*, T^*)$ is second countable and compact Hausdorff. Since for compact Hausdorff spaces, second countability is equivalent to metrizability, $(X^*, T^*)$ is metrizable. Therefore, as a subspace of a metric space, $(X, T)$ is also a metrizable topological space.