

Midterm 2 – Math 5080 – Spring 2004

Name: _____

Instructions. READ CAREFULLY.:

- (i) The work you turn in must be your own. You may not discuss the midterm with anyone, either in the class or outside the class. [You may of course consult with me for clarification of any of the problems.] *Failure to follow this policy will be considered cheating and will result in a course grade of E.*
- (ii) You may consult the textbook and your notes. In particular, feel free to use any of the information in the tables of distributions in Appendix B, for example, the moment generating functions for specific distributions. You can use a **general mathematical reference**, for example a calculus text or a table of integrals. You *may not* use any statistical textbook or written source material concerning the specific subject matter of the course. *Failure to follow this policy will be considered cheating and will result in a course grade of E.*
- (iii) Your midterm must be clearly written and legible. **I will not grade problems which are sloppily presented and such problems will receive a grade of 0.** If you are unable to write legibly and clearly, use of a word processor. You have 7 days to complete the midterm; budget time for writing up your solutions.
- (iv) Think about your exposition. Someone (me) has to read what you have written. Your answer is only correct if I can understand what you have done. Style matters.
- (v) Midterms are due at 6 PM on Wednesday April 21, 2004. **Late midterms will not be accepted**, except for reasons of death or serious illness.

Sign here to indicate you have read and understand these instructions.

There are 6 problems total, with a point total of 75. Note the problems are not all worth the same amount.

Problem 1 (10 points). Let X_1, X_2, \dots, X_n be a random sample from a Normal($\mu = 3, \sigma^2 = 4$) distribution.

(i) How big must n be so that the probability \bar{X} is within 0.01 of 3 is exactly 0.95?

(ii) When $n = 10$, what is $P(\sum_{i=1}^{10} (X_i - 3)^2 > 40)$?

Solution. \bar{X} has a Normal distribution with mean 3 and variance $4/n$.

$$\begin{aligned} P(-0.01 < \bar{X} - 3 < 0.01) &= P\left(\frac{-0.01}{2/\sqrt{n}} < \frac{\bar{X} - 3}{2/\sqrt{n}} < \frac{0.01}{2/\sqrt{n}}\right) \\ &= P(-z < Z < z) , \end{aligned}$$

where Z is a standard Normal r.v. and $z = \sqrt{n}0.005$.

Thus, since we want

$$P(-z < Z < z) = 0.95 ,$$

We should take $z = 1.96$, using a table of the standard normal distribution. Solving for n in

$$1.96 = \sqrt{n}0.005$$

gives $n = 153664$.

For the second part,

$$\begin{aligned} P\left(\sum_{i=1}^{10} (X_i - 3)^2 > 40\right) &= P\left(\sum_{i=1}^{10} \left(\frac{X_i - 3}{2}\right)^2 > 10\right) \\ &= P(\chi_{10}^2 > 10) , \end{aligned}$$

where χ_{10}^2 is a r.v. with a Chi-Squared distribution with 10 degrees of freedom. Using a table of a Chi-Squared distribution, this probability must be 0.44. \square

Problem 2 (5 points). Let X_1, X_2, \dots, X_{10} be a random sample from a $\text{Normal}(\mu, \sigma^2)$ distribution, and Y_1, \dots, Y_{11} be a random sample from a $\text{Normal}(\nu, \tau^2)$ distribution, independent of the first sample. Let S_X^2 and S_Y^2 denote the sample variances for the two samples. Compute $P(S_X/S_Y > 1.533\frac{\sigma}{\tau})$.

Solution. We have

$$P\left(\frac{S_X}{S_Y} > 1.533\frac{\sigma}{\tau}\right) = P\left(\frac{S_X^2/\sigma^2}{S_Y^2/\tau^2} > 1.533^2\right).$$

The r.v. $F_{9,10} = \frac{\tau^2 S_X^2}{\sigma^2 S_Y^2}$ has a F distribution with 9 and 10 degrees of freedom. Using a table, we see that

$$P(F_{9,10} > 2.35) \approx 1 - 0.90 = 0.10.$$

□

Problem 3 (10 points). Let X_1, \dots, X_n be a random sample from the double exponential distribution:

$$f(x; \theta, \eta) = \frac{1}{2\theta} e^{-|x-\eta|/\theta}.$$

- (i) Find the method of moments estimator of (θ, η) .
- (ii) Show that they are consistent.

Solution. The mean and variance of a double exponential distribution are η and $2\theta^2$ respectively. We have

$$E(X_1^2) = \text{Var}(X_1) + (E(X_1))^2 = 2\theta^2 + \eta^2.$$

Thus the method of moment equations are

$$\begin{aligned} \eta &= \bar{X}, \\ 2\theta^2 + \eta^2 &= \overline{X^2}, \end{aligned}$$

where $\overline{X^2} = n^{-1} \sum_{i=1}^n X_i^2$ is the second sample moment. Solving for η and θ we have

$$\begin{aligned} \hat{\eta} &= \bar{X} \\ \hat{\theta} &= \sqrt{\frac{1}{2} (\overline{X^2} - (\bar{X})^2)}. \end{aligned}$$

□

By the Weak Law of Large Numbers, we have that

$$\hat{\eta} = \bar{X} \xrightarrow{Pr} E(X_1) = \eta.$$

[Here \xrightarrow{Pr} denotes convergence in probability.]

Also by the Weak Law of Large Numbers,

$$\overline{X^2} \xrightarrow{Pr} E(X_1^2) = 2\theta^2 + \eta^2.$$

Then it follows that

$$\hat{\theta} \rightarrow \sqrt{\frac{1}{2} (2\theta^2 + \eta^2 - \eta^2)} = \theta.$$

This shows that $\hat{\eta}$ and $\hat{\theta}$ are consistent estimators.

Problem 4 (10 points). Suppose a bit (a 0 or 1) is sent from a transmitter to n receivers over a noisy channel. Each receiver registers the correct bit (the bit sent by the transmitter) with probability 0.95, and the incorrect bit with probability 0.05. The n receivers are independent of one another. Suppose that n is odd. Find the maximum likelihood estimate of the bit which was sent, where the data is the n received bits.

Hint: The parameter here is the sent bit $\rho \in \{0, 1\}$. Write down the probability mass function for the received bits given ρ .

Solution.

$$\begin{aligned}
 P(X_1 = x_1, \dots, X_n = x_n \mid \rho = 1) &= \prod_{i=1}^n (0.95)^{x_i} (0.05)^{1-x_i} \\
 &= (0.95)^{\sum x_i} (0.05)^{n-\sum x_i} \\
 &= \left(\frac{0.95}{0.05}\right)^{\sum x_i} (0.05)^n \\
 P(X_1 = x_1, \dots, X_n = x_n \mid \rho = 0) &= \prod_{i=1}^n (0.95)^{1-x_i} (0.05)^{x_i} \\
 &= (0.95)^{n-\sum x_i} (0.05)^{\sum x_i} \\
 &= \left(\frac{0.95}{0.05}\right)^{-\sum x_i} (0.95)^n
 \end{aligned}$$

We pick as our estimate of ρ the value which makes the probability

$$P(X_1 = x_1, \dots, X_n = x_n \mid \rho)$$

as large as possible. Thus we choose $\hat{\rho} = 1$ if and only if (otherwise choosing $\hat{\rho} = 0$)

$$\begin{aligned}
 \left(\frac{0.95}{0.05}\right)^{\sum x_i} (0.05)^n &> \left(\frac{0.95}{0.05}\right)^{-\sum x_i} (0.95)^n \\
 \left(\frac{0.95}{0.05}\right)^{2\sum x_i} &> \left(\frac{0.95}{0.05}\right)^n
 \end{aligned}$$

This holds if and only if (taking logarithms)

$$\sum x_i > \frac{n}{2},$$

i.e. if and only if there are more ones received than zeros. Since n is odd, there are never any “ties”. □

Problem 5 (20 points). Let X_1, \dots, X_n be a random sample from the distribution with the following density

$$f(x; \lambda) = \lambda(1+x)^{-(1+\lambda)} \mathbf{1}\{x \geq 0\} .$$

- (i) Find the MLE $\hat{\lambda}$ for λ .
- (ii) Is $\hat{\lambda}$ unbiased for λ ? If not, find a function of $\hat{\lambda}$ which is an unbiased estimator of λ .
- (iii) Compute the Cramer-Rao Lower Bound for unbiased estimators of λ .
- (iv) Find the variance of the unbiased estimator found above. Compare to the Cramer-Rao Lower Bound.

Hint: $\log(1 + X_i)$ has a familiar distribution.

Solution. We first find $\hat{\lambda}$. The likelihood and log-likelihood functions are

$$L(\lambda; \mathbf{x}) = \lambda^n \prod_{i=1}^n (1+x_i)^{-1-\lambda}$$

$$\ell(\lambda; \mathbf{x}) = n \log \lambda - (1+\lambda) \sum_{i=1}^n \log(1+x_i) .$$

We solve for critical points:

$$0 = \frac{d}{d\lambda} \ell(\lambda; \mathbf{x})$$

$$= \frac{n}{\lambda} - \sum_{i=1}^n \log(1+x_i) .$$

Thus,

$$\lambda = \left[\frac{1}{n} \sum_{i=1}^n \log(1+x_i) \right]^{-1}$$

is a critical point, and it can be checked that it is a global maximum. We conclude that the MLE is

$$\hat{\lambda} = \left[\frac{1}{n} \sum_{i=1}^n \log(1+X_i) \right]^{-1} .$$

Now, let $h(x) = \log(1 + x)$, and let $Y = h(X)$. Notice that $h^{-1}(y) = e^y - 1$.

$$\begin{aligned} f_Y(y) &= f_X(h^{-1}(y)) \frac{d}{dy} h^{-1}(y) \\ &= \lambda (e^y)^{-\lambda-1} e^y \\ &= \lambda e^{-\lambda y}. \end{aligned}$$

Thus $Y_i = \log(1 + X_i)$ has an Exponential($1/\lambda$) distribution. Then $S_n = \sum_{i=1}^n \log(1 + X_i)$ has a Gamma(n, λ^{-1}) distribution, and $W = 2\lambda S_n$ has a Chi-Squared($2n$) distribution. We can compute $E(W^{-1})$ and $\text{Var}(W^{-1})$:

$$\begin{aligned} E(W^{-1}) &= \int_0^{\infty} \frac{1}{w} \frac{1}{2^{\frac{2n}{2}} (\frac{2n}{2} - 1)!} w^{\frac{2n}{2}-1} e^{-\frac{1}{\lambda} w} dw \\ &= \frac{1}{2(n-1)} \int_0^{\infty} \frac{1}{2^{\frac{2n-2}{2}} (\frac{2n-2}{2} - 1)!} w^{\frac{2n-2}{2}-1} e^{-\frac{1}{\lambda} w} dw \\ &= \frac{1}{2(n-1)} \end{aligned}$$

and

$$\begin{aligned} E(W^{-2}) &= \int_0^{\infty} \frac{1}{w^2} \frac{1}{2^{\frac{2n}{2}} (\frac{2n}{2} - 1)!} w^{\frac{2n}{2}-1} e^{-\frac{1}{\lambda} w} dw \\ &= \frac{1}{4(n-1)(n-2)} \int_0^{\infty} \frac{1}{2^{\frac{2n-4}{2}} (\frac{2n-4}{2} - 1)!} w^{\frac{2n-4}{2}-1} e^{-\frac{1}{\lambda} w} dw \\ &= \frac{1}{4(n-1)(n-2)}. \end{aligned}$$

We conclude that $\text{Var}(W^{-1}) = \frac{1}{4(n-1)^2(n-2)}$. Since $\hat{\lambda} = \frac{2\lambda n}{W}$, we have

$$E(\hat{\lambda}) = \frac{2\lambda n}{2(n-1)} = \lambda \frac{n}{n-1}.$$

We conclude that $\hat{\lambda}$ is a biased estimator. It is easy to adjust it to get an unbiased estimator: Let $\tilde{\lambda} = \frac{n-1}{n} \hat{\lambda} = (n-1)/S_n$; it satisfies $E(\tilde{\lambda}) = \lambda$ and consequently is unbiased.

Let us compute the information:

$$\begin{aligned} I_1(\lambda) &= E_\lambda \left(-\frac{d^2}{d\lambda^2} [\log \lambda - (1 + \lambda) \log(1 + X_1)] \right) \\ &= E_\lambda \left(-\frac{d}{d\lambda} \left[\frac{1}{\lambda} - \log(1 + X_1) \right] \right) \\ &= E_\lambda \left(\frac{1}{\lambda^2} \right) \\ &= \frac{1}{\lambda^2} \end{aligned}$$

For the entire sample, $I(\lambda) = \frac{n}{\lambda^2}$, and the Cramer-Rao lower bound is λ^2/n .

$$\begin{aligned} \text{Var}(\tilde{\lambda}) &= \text{Var} \left(\frac{2\lambda(n-1)}{W} \right) \\ &= 4(n-1)^2 \lambda^2 \frac{1}{4(n-1)^2(n-2)} \\ &= \frac{\lambda^2}{n-2}. \end{aligned}$$

□

Problem 6 (20 points). Suppose an experiment can result in 3 possible outcomes. For given values α, β , these outcomes have probabilities

outcome	I	II	III
probability	α	2α	$2\alpha + \beta$

Suppose that n independent experiments are conducted, and N_i outcomes of type i are observed, for $i = 1, 2, 3$. The parameters α and β are such that the table above gives a probability distribution over the three possible outcomes. Find the MLEs of α and β based on this data.

Solution. Notice that $\alpha + 2\alpha + 2\alpha + \beta = 1$, so $\beta = 1 - 5\alpha$. We have then that in terms of α , the three probabilities are

$$\alpha, 2\alpha, 1 - 3\alpha.$$

Notice that we must have $\alpha \in [0, 1/3]$ for these to be non-negative and less than one. The likelihood function is

$$L(\alpha; n_1, n_2, n_3) = \binom{n}{n_1 n_2 n_3} (\alpha)^{n_1} (2\alpha)^{n_2} (1 - 3\alpha)^{n_3}$$

$$\ell(\alpha) = (n_1 + n_2) \log \alpha + n_3 \log(1 - 3\alpha) + \log \binom{n}{n_1 n_2 n_3} + n_2 \log 2$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n_1 + n_2}{\alpha} - \frac{3n_3}{1 - 3\alpha}$$

Setting $\frac{\partial \ell}{\partial \alpha} = 0$ and solving for the critical point yields

$$0 = \frac{n_1 + n_2}{\alpha} - \frac{3n_3}{1 - 3\alpha}$$

$$0 = (1 - 3\alpha)(n_1 + n_2) - 3\alpha n_3$$

$$0 = n_1 + n_2 - 3\alpha(n_1 + n_2 + n_3)$$

$$\alpha = \frac{n_1 + n_2}{3(n_1 + n_2 + n_3)}.$$

It can be checked that this is indeed a global maximum. Thus $\hat{\alpha} = (n_1 + n_2)/3n$. Note that since $n_1 + n_2 \leq n$, we always have $\hat{\alpha} \in [0, \frac{1}{3}]$, as it should be. Since $\beta = 1 - 3\alpha$, we

have

$$\hat{\beta} = 1 - 3 \frac{n_1 + n_2}{3(n_1 + n_2 + n_3)}.$$

□