

Midterm 1 – Math 5080 – Spring 2004

Name: _____

Instructions. READ CAREFULLY.:

- (i) The work you turn in must be your own. You may not discuss the midterm with anyone, either in the class or outside the class. [You may of course consult with me for clarification of any of the problems.] *Failure to follow this policy will be considered cheating and will result in a course grade of E.*
- (ii) You may consult the textbook and your notes. In particular, feel free to use any of the information in the tables of distributions in Appendix B, for example, the moment generating functions for specific distributions. You *may not* use any other written source. *Failure to follow this policy will be considered cheating and will result in a course grade of E.*
- (iii) Your midterm must be clearly written and legible. **I will not grade problems which are sloppily presented and such problems will receive a grade of 0.** If you are unable to write legibly and clearly, use of a word processor. You have 7 days to complete the midterm; budget time for writing up your solutions.
- (iv) Think about your exposition. Someone (me) has to read what you have written. Your answer is only correct if I can understand what you have done. Style matters.
- (v) Midterms are due at 6 PM on Wednesday Feb 25, 2003. **Late midterms will not be accepted**, except for reasons of death or serious illness.

Sign here to indicate you have read and understand these instructions.

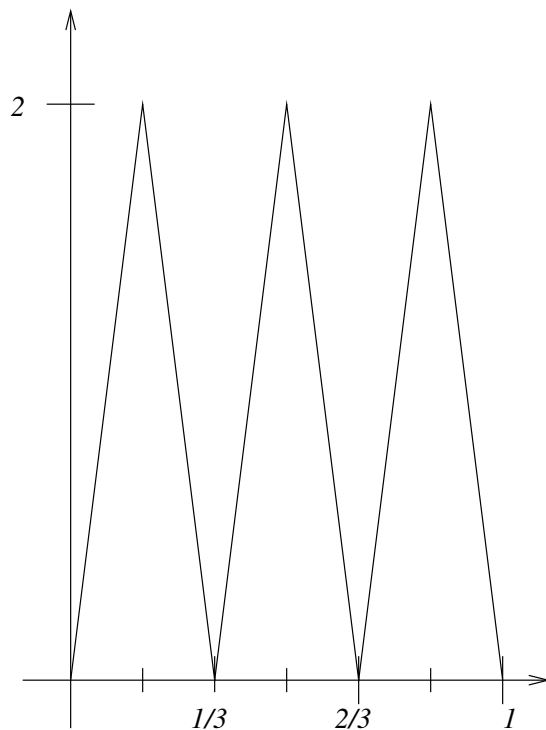
Problem 1. Let X be a uniform random variable on $[0, 1]$. Consider the function $g : [0, 1] \rightarrow \mathbb{R}$ defined by

$$g(x) = \begin{cases} 12x & \text{if } 0 \leq x < 1/6 \\ 4 - 12x & \text{if } 1/6 \leq x < 1/3 \\ 12x - 4 & \text{if } 1/3 \leq x < 1/2 \\ 8 - 12x & \text{if } 1/2 \leq x < 2/3 \\ 12x - 8 & \text{if } 2/3 \leq x < 5/6 \\ 12 - 12x & \text{if } 5/6 \leq x < 1. \end{cases}$$

Find the pdf of $Y = g(X)$.

Hint: Sketch the function g .

Solution.



Let $y \in (0, 2)$. Notice that $g(x) > y$ if and only if $y/12 < x < (4-y)/12$, or $(4+y)/12 <$

$x < (8 - y)/12$, or if $(8 + y)/12 < x < (12 - y)/12$. Thus

$$\begin{aligned} P(Y > y) &= P(g(X) > y) \\ &= P\left(X \in \left(\frac{y}{12}, \frac{4-y}{12}\right) \cup \left(\frac{4+y}{12}, \frac{8-y}{12}\right) \cup \left(\frac{8+y}{12}, \frac{12-y}{12}\right)\right) \\ &= \frac{3(4-2y)}{12} \\ &= 1 - \frac{y}{2}. \end{aligned}$$

We conclude that the density of Y is

$$f_Y(y) = \begin{cases} \frac{1}{2} & \text{if } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}.$$

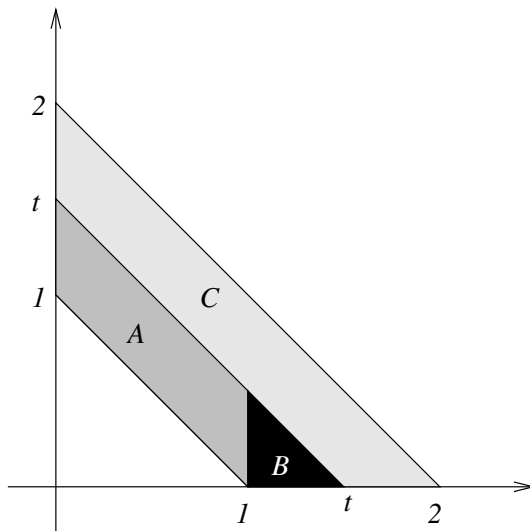
□

Problem 2. Let X and Y have a joint pdf

$$f(x, y) = \begin{cases} \frac{8}{5}xy & \text{if } x > 0, y > 0, 1 < x + y < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find the pdf of $X + Y$.

Solution. The region where f is positive is given by $A \cup B \cup C$ in the figure below.



We have

$$P(X + Y \leq t) = P((X, Y) \in R_t),$$

where

$$R_t = \{(x, y) : y < t - x, x > 0, y > 0\}.$$

Notice that $R_t = A \cup B$ in the figure shown. Thus for $1 \leq t \leq 2$

$$\begin{aligned} P(X + Y \leq t) &= \iint_{A \cup B} \frac{8}{5} xy dx dy \\ &= \iint_A \frac{8}{5} xy dx dy + \iint_B \frac{8}{5} xy dx dy \\ &= \int_0^1 \int_{1-x}^{t-x} \frac{8}{5} xy dy dx + \int_1^t \int_0^{t-x} \frac{8}{5} xy dy dx \\ &= \frac{8}{5} \left\{ \int_0^1 x \left[\int_{1-x}^{t-x} y dy \right] dx + \int_1^t x \left[\int_0^{t-x} y dy \right] dx \right\} \\ &= \frac{8}{5} \left\{ \int_0^1 x \left[\frac{1}{2} y^2 \Big|_{1-x}^{t-x} \right] dx + \int_1^t x \left[\frac{1}{2} y^2 \Big|_0^{t-x} \right] dx \right\} \\ &= \frac{8}{5} \left\{ \int_0^1 \left(\frac{(t^2 - 1)}{2} x - (t - 1)x^2 \right) dx + \int_1^t \left(\frac{t^2}{2} x - tx^2 + \frac{1}{2} x^3 \right) dx \right\} \\ &= \frac{8}{5} \left\{ \left[\frac{(t^2 - 1)}{4} x^2 - \frac{(t - 1)}{3} x^3 \right]_0^1 + \left[\frac{t^2}{4} x^2 - \frac{t}{3} x^3 + \frac{1}{8} x^4 \right]_1^t \right\} \\ &= \frac{8}{5} \left\{ \frac{(t - 1)(3t - 1)}{12} + \frac{t^4 - 6t^2 + 8t - 3}{24} \right\} \\ &= \frac{t^4 - 1}{15} \end{aligned}$$

Thus we have found the cdf for $X + Y$. By differentiating we have

$$f_{X+Y}(t) = \begin{cases} \frac{4}{15}t^3 & \text{if } 1 \leq t \leq 2, \\ 0 & \text{if } t < 1 \text{ or } t > 2. \end{cases}$$

□

Problem 3. Let X, Y be independent $\text{Normal}(0, 1)$ random variables.

(a) What is the joint pdf of $R = X^2 + Y^2$ and $\Theta = \arctan(Y/X)$?

(b) What are the marginal distributions of R and Θ ? Are R and Θ independent?

Solution. Let $h(x, y) = (x^2 + y^2, \arctan(y/x))$. Setting $(r, \theta) = h(x, y)$ and solving for x and y in terms of r and θ gives

$$h^{-1}(r, \theta) = (\sqrt{r} \cos \theta, \sqrt{r} \sin \theta).$$

Thus

$$Dh^{-1}(r, \theta) = \begin{bmatrix} \frac{1}{2\sqrt{r}} \cos \theta & -\sqrt{r} \sin \theta \\ \frac{1}{2\sqrt{r}} \sin \theta & \sqrt{r} \cos \theta \end{bmatrix},$$

and so

$$J_{h^{-1}}(r, \theta) = \frac{1}{2} \cos^2 \theta + \frac{1}{2} \sin^2 \theta = \frac{1}{2}.$$

Thus the joint pdf of (R, Θ) is, for $0 \leq \theta < 2\pi$ and $r > 0$,

$$f_{R,\Theta}(r, \theta) = \frac{1}{2\pi} e^{-\frac{1}{2}r} \frac{1}{2} = \left(\frac{1}{2} e^{-\frac{r}{2}} \right) \left(\frac{1}{2\pi} \right).$$

Since $f_{R,\Theta}(r, \theta)$ factors, we conclude that R and Θ are independent, and moreover R is an $\text{Exponential}(1/2)$ r.v. and Θ a $\text{Uniform}(0, 2\pi)$ r.v. □

Problem 4. Let X and Y be independent $\text{Normal}(0, 1)$ random variables.

(i) Find the distribution of $U = XY^{-1}$.

(ii) What is the expectation of U ?

Solution. Introduce the r.v. $V = X$, and consider the pair $(U, V) = h(X, Y)$, where $h(x, y) = (xy^{-1}, x)$. Then $h^{-1}(u, v) = (v, vu^{-1})$, and

$$Dh^{-1}(u, v) = \begin{bmatrix} 0 & 1 \\ -vu^{-2} & u^{-1} \end{bmatrix}.$$

Thus the Jacobian is $J_{h^{-1}} = |vu^{-2}|$, and

$$f_{U,V} = \frac{1}{2\pi} e^{-\frac{1}{2}(v^2 + v^2 u^{-2})} |v| u^{-2} = \frac{1}{2\pi} \frac{1}{u^2} |v| e^{-\frac{1}{2}(1+u^{-2})v^2}.$$

Now we integrate out V :

$$\begin{aligned} f_U(u) &= \frac{1}{2\pi} \frac{1}{u^2} \int_{-\infty}^{\infty} |v| e^{-\frac{1}{2}(1+u^{-2})v^2} dv \\ &= \frac{1}{2\pi} \frac{1}{u^2} 2 \int_0^{\infty} v e^{-\frac{1}{2}(1+u^{-2})v^2} dv \\ &= \frac{1}{\pi} \frac{1}{u^2} \frac{1}{1+u^{-2}} \\ &= \frac{1}{\pi} \frac{1}{1+u^2}. \end{aligned}$$

This distribution has a name: it is called the *Cauchy* distribution.

Notice that

$$\int_{\mathbb{R}} |u| \frac{1}{\pi(1+u^2)} dy \geq 2 \int_1^{\infty} \frac{u}{\pi(1+u^2)} du = \infty,$$

so the expectation is infinite. □

Problem 5. Let X_1, X_2, X_3, X_4 be independent and identically distributed random variables, each with a Uniform $[0, 1]$ distribution. Let Y_1, Y_2, Y_3, Y_4 denote the order statistics for this random sample. Find the pdf for $Y_3 - Y_2$.

Solution. The joint pdf of (Y_2, Y_3) is given by

$$f_{Y_2, Y_3}(y_2, y_3) = 4! y_2 (1 - y_3) \mathbf{I}\{0 \leq y_2 < y_3 \leq 1\}.$$

Let $U = Y_3 - Y_2$ and $V = Y_2$. Then $Y_2 = V$ and $Y_3 = U + V$. It is easy to check (as we have done this before!) that the Jacobian of the inverse transform is 1. Thus

$$f_{U,V}(u,v) = 4!v(1-u-v)\mathbf{I}\{0 \leq v < u+v \leq 1\}.$$

Integrating out v gives

$$\begin{aligned} \int_0^{1-u} 4!v(1-u-v)dv &= 24 \int_0^{1-u} [(1-u)v - v^2]dv \\ &= 24 \left[\frac{(1-u)}{2}v^2 - \frac{1}{3}v^3 \right]_0^{1-u} \\ &= 4(1-u)^3. \end{aligned}$$

□

Problem 6. Let X and Y be independent $\text{Normal}(0, 1)$ random variables. What is the joint distribution of $\frac{X-Y}{\sqrt{2}}$ and $\frac{X+Y}{\sqrt{2}}$?

Hint: What kind of transformation is $(x, y) \mapsto ((x-y)/\sqrt{2}, (x+y)/\sqrt{2})$?

Solution. This is an orthonormal transformation, so the transformed variables are also independent standard Normal r.v.s. □

Problem 7. If X_1, X_2, \dots, X_n are i.i.d. $\text{Exponential}(\lambda)$ random variables, what is the distribution of $X_1 + \dots + X_n$? Justify your answer.

Solution. Check that the MGF of $X_1 + \dots + X_n$ is the MGF for a Gamma random variable. □

Problem 8. Let X_n be a negative Binomial r.v. with parameters r and $p = n/\lambda$. Find a random variable X so that $\frac{1}{n}X_n \xrightarrow{d} X$. Justify your answer.

Hint: What is the relationship of the negative binomial to the Geometric? Recall the limit law we proved in class for the Geometric.

Solution. We know that the MGF for X_n is

$$M_{X_n}(t) = \left(\frac{\lambda n^{-1} e^t}{1 - (1 - \lambda n^{-1}) e^t} \right)^r .$$

so that

$$\begin{aligned} M_{n^{-1}X_n}(t) &= \left(\frac{\lambda n^{-1} e^{t/n}}{1 - (1 - \lambda n^{-1}) e^{t/n}} \right)^r \\ &= \left(\frac{\lambda n^{-1} (1 + tn^{-1} + O(n^{-2}))}{1 - (1 - \lambda n^{-1})(1 + tn^{-1} + O(n^{-2}))} \right)^r \\ &= \left(\frac{\lambda n^{-1} + O(n^{-2})}{1 - (1 + tn^{-1} - \lambda n^{-1} + O(n^{-2}))} \right)^2 \\ &= \left(\frac{\lambda + O(n^{-1})}{\lambda - t + O(n^{-1})} \right)^r . \end{aligned}$$

We conclude that

$$\lim_{n \rightarrow \infty} M_{n^{-1}X_n}(t) = \left(\frac{1}{1 - \lambda^{-1}t} \right)^r .$$

That is, $n^{-1}X_n \xrightarrow{d} X$, where X is Gamma with parameters $\kappa = r$ and $\theta = \lambda^{-1}$. □

Problem (Extra Credit). Let X_n be a Binomial($n, 1/2$) r.v. Prove directly, without using the Central Limit Theorem, that if $Y_n = n^{-1/2}(X_n - \frac{n}{2})$, then $Y_n \xrightarrow{d} N$, where N is Normal($0, \frac{1}{4}$).

Solution. Some expansions which are useful are

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + O(x^3) \\ \log(1+x) &= x - \frac{x^2}{2} + O(x^3), \end{aligned}$$

provided x is in a neighborhood of 0. [Here the notation $f(x) = O(g(x))$ means that $|f(x)/g(x)| < \kappa$ for some constant κ not depending on x .]

We have

$$M_{X_n}(t) = \left(\frac{1}{2} e^t + \frac{1}{2} \right)^n .$$

Thus,

$$\begin{aligned}
M_{Y_n}(t) &= E\left(e^{tn^{-1/2}(Y_n - 2^{-1}n)}\right) \\
&= e^{-t2^{-1}n^{1/2}} E\left(e^{(tn^{-1/2})Y_n}\right) \\
&= e^{-t2^{-1}n^{1/2}} M_{Y_n}(tn^{-1/2}) \\
&= e^{-t2^{-1}n^{1/2}} \left(\frac{1}{2}e^{tn^{-1/2}} + \frac{1}{2}\right)^n
\end{aligned}$$

Taking logarithms on both sides gives

$$\log M_{Y_n}(t) = -\frac{t\sqrt{n}}{2} + n \log\left(\frac{1}{2}e^{t/\sqrt{n}} + \frac{1}{2}\right). \quad (1)$$

Now using a Taylor expansion on e^x gives

$$\frac{1}{2}e^{t/\sqrt{n}} = \frac{1}{2} + \frac{t}{2\sqrt{n}} + \frac{t^2}{4n} + O(n^{-3/2}).$$

So

$$n \log\left(\frac{1}{2}e^{t/\sqrt{n}} + \frac{1}{2}\right) = n \log\left(1 + \frac{t}{2\sqrt{n}} + \frac{t^2}{4n} + O(n^{-3/2})\right). \quad (2)$$

Using a Taylor expansion of $\log(1+x)$ and (2)

$$\begin{aligned}
n \log\left(\frac{1}{2}e^{t/\sqrt{n}} + \frac{1}{2}\right) &= n \left[\frac{t}{2\sqrt{n}} + \frac{t^2}{4n} + O(n^{-3/2}) - \frac{t^2}{8n} + O(n^{-3/2}) \right] \\
&= \frac{t\sqrt{n}}{2} + \frac{t^2}{8} + O(n^{-1/2}).
\end{aligned} \quad (3)$$

Now using (3) in (1) gives

$$\log M_{Y_n}(t) = \frac{t^2}{8} + O(n^{-1/2}),$$

and letting $n \rightarrow \infty$ shows that

$$\lim_{n \rightarrow \infty} M_{Y_n}(t) = e^{\frac{1}{2} \frac{1}{4} t^2}.$$

We conclude that Y_n converges in distribution to a $N(0, \frac{1}{4})$ random variable. \square