Exercise 2.28 There are several ways to do this. First notice that what we want is the *conditional* probability of $O_1 = 0, O_2 = 1$ given that $O_1 \neq O_2$, since we only consider the outcomes of the two tosses when $O_1 \neq O_2$. So we want compute:

$$\begin{aligned}
\mathbb{P}\{O_1 = 0, O_1 = 1 \mid O_1 \neq O_2\} &= \frac{\mathbb{P}\{O_1 = 0, O_1 = 1 \text{ and } O_1 \neq O_2\}}{\mathbb{P}\{O_1 \neq O_2\}} \\
&= \frac{p(1-p)}{2p(1-p)} \\
&= \frac{1}{2}\,.
\end{aligned}$$

Alternatively, we can reason as follows: Let T be the number of trials required before the two coins come up differently.

$$\begin{aligned}
\mathbb{P}\{X = 1\} &= \sum_{n=1}^{\infty} \mathbb{P}\{X = 1 \text{ and } T = n\} \\
&= \sum_{m=0}^{\infty} (p^2 + (1-p)^2)^m p(1-p) \\
&= \frac{p(1-p)}{1 - (p^2 + (1-p)^2)} \\
&= \frac{1}{2}\,.
\end{aligned}$$

The proceedure in (b) is to flip a coin until the last flip is different from the next to last flip. Equivalently, flip until the coin is different from the first flip. With probability p this will be tails, and with probability $1-p$ this will be heads.

Exercise 2.34 $f(x)$ must integrate to 1, so

$$\begin{aligned}
\int_0^2 c(4x - 2x^2)dx &= c(2x^2 - \frac{2}{3}x^3)\Big|_0^2 \\
&= c(8 - \frac{16}{3}) \\
&= c\frac{8}{3}\,.
\end{aligned}$$

Thus $c = \frac{3}{8}$, and

$$f(x) = \frac{3}{2}x - \frac{3}{4}x^2\,.$$

$$
\begin{aligned}
\mathbb{P}\left\{\frac{1}{2} < X < \frac{3}{2}\right\} &= \int_{1/2}^{3/2} \left(\frac{3}{2}x - \frac{3}{4}x^2\right) dx \\
&= \frac{3}{4}x^2 - \frac{1}{4}x^3 \Big|_{1/2}^{3/2} \\
&= \frac{5}{4}
\end{aligned}
$$

Exercise 2.37

$$
\begin{aligned}
\mathbb{P}\left\{\max_{1\le k\le n} X_k \le x\right\} &= \mathbb{P}\left\{X_1 \le x, X_2 \le x, \dots, X_n \le x\right\} \\
&= \mathbb{P}\left\{X_1 \le x\right\} \cdots \mathbb{P}\left\{X_n \le x\right\} \\
&= x^n .
\end{aligned}
$$

Since the distribution function is $F_M(x)$, the density is given by

$$
f_M(x) = \frac{d}{dx} F_M(x) = nx^{n-1} .
$$

Exercise 2.43

$$
X = \sum_{i=1}^{n} X_i .
$$

Thus

$$
\begin{aligned}
\mathbb{E}\{X\} &= \sum_{i=1}^{n} \mathbb{E}\{X_i\} \\
&= n\mathbb{P}\left\{\text{red ball 1 is take before a black ball}\right\}
\end{aligned}
$$

The full solution will be provided later. See the current computer homework assignment.

Example 2.44

Let

$$
Y_i = \begin{cases} 1 & \text{if red ball } i \text{ is chosen between first and second black ball drawn} \\ 0 & \text{otherwise} \end{cases} .
$$

As before, then

$$\mathbb{E}\{Y\} = n\mathbb{P}\{\text{Red ball 1 is between first and second black balls}\}\,.$$

The full solution is part of the next computer HW.

Exercise 2.47

$$\mathbb{E}\left\{X^2\right\} = \int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}\,.$$

Exercise 2.53

$$\mathbb{E}\{X^n\} = \int_0^1 x^n dx = \frac{1}{n+1}x^{n+1}\Big|_0^1 = \frac{1}{n+1}\,.$$

Also,

$$\mathbb{E}\left\{(X^n)^2\right\} = \int_0^1 x^{2n} dx = \frac{1}{2n+1}\,.$$

Thus,

$$\begin{aligned}\mathrm{Var}(X^n) &= \mathbb{E}\left\{(X^n)^2\right\} - (\mathbb{E}\{X^n\})^2 \\ &= \frac{1}{2n+1} - \left(\frac{1}{n+1}\right)^2\,.\end{aligned}$$

Exercise 2.57 X is the sum of n Bernoulli(p) r.v.s, and Y is the sum of m Bernoulli(p) r.v.s, and so $X+Y$ is the sum of $n+m$ Bernoulli(p) r.v.s.

Exercise 2.67

$$\begin{aligned}\mathbb{P}\{5 < X < 15\} &= \mathbb{P}\{-5 < X - 10 < 5\} \\ &= \mathbb{P}\left\{(X-10)^2 < 25\right\} \\ &\le \frac{\mathrm{Var}(X)}{25} \\ &= \frac{3}{5}\,.\end{aligned}$$

Exercise 3.12 The marginal of Y is given by

$$\begin{aligned} f_Y(y) &= \int_0^\infty \frac{e^{-x/y}e^{-y}}{y}dx \\ &= \frac{e^{-y}}{y}\int_0^\infty e^{-x/y}dx \\ &= \frac{e^{-y}}{y}\, ye^{-x/y}\Big|_0^\infty \\ &= \frac{e^{-y}}{y}y \\ &= e^{-y}\,. \end{aligned}$$

Thus the conditional density of $X \mid Y = y$ is given by

$$\frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{y}e^{-x/y}\,.$$

That is to say, given $Y = y$, the distribution of X is exponential$(1/y)$. Thus it has conditional expectation y.

Exercise 3.14 Notice that for $u \leq 1/2$,

$$\begin{aligned} \mathbb{P}\left\{U \leq u \mid U \leq \frac{1}{2}\right\} &= \frac{\mathbb{P}\left\{U \leq u \text{ and } u \leq \frac{1}{2}\right\}}{\mathbb{P}\left\{U \leq \frac{1}{2}\right\}} \\ &= 2\mathbb{P}\left\{U \leq u\right\} \\ &= 2u\,. \end{aligned}$$

Thus the density of U conditioned on $U < 1/2$ is

$$f(u) = \begin{cases} 2 & \text{if } 0 < u < 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

And so

$$\begin{aligned} \mathbb{E}\left\{U \mid U < \frac{1}{2}\right\} &= \int_0^{1/2} 2u du \\ &= u^2\Big|_0^{1/2} \\ &= \frac{1}{4}\,. \end{aligned}$$

Exercise 3.30

$$
\begin{aligned}
\mathbb{E}\{N\} &= \sum_{j=1}^{m} \mathbb{E}\{N \mid X_0 = j\}\, p(j) \\
&= \sum_{j=1}^{m} \frac{1}{p(j)} p(j) \\
&= m\,.
\end{aligned}
$$

Thus follows because conditional on $X_0 = j$, the distribution of N is geometric$(p(j))$.

Exercise 3.36

$$
\begin{aligned}
\mathbb{E}\{N\} &= \mathbb{E}\{\mathbb{E}\{N \mid U\}\} \\
&= \mathbb{E}\{nU\} \\
&= \frac{n}{2}\,, \\
\mathbb{E}\left\{N^2\right\} &= \mathbb{E}\left\{\mathbb{E}\left\{N^2 \mid U\right\}\right\} \\
&= \mathbb{E}\left\{nU(1-U) + (nU)^2\right\} \\
&= \mathbb{E}\left\{nU + n(n-1)U^2\right\} \\
&= \frac{n}{2} + n(n-1)\frac{1}{3} \\
\mathrm{Var}(N) &= \frac{n}{2} + n(n-1)\frac{1}{3} - \frac{n^2}{4} \\
&= \frac{1}{6}n + \frac{1}{12}n^2\,.
\end{aligned}
$$

Exercise 3.42 If A is the total amount spend, then $N = \sum_{i=1}^{N} U_i$ where $\{U_i\}$ is an i.i.d. sequence of Uniform$[0, 100]$ r.v.s., and N is Poisson(10).

Thus

$$
\begin{aligned}
\mathbb{E}\{A\} &= \mathbb{E}\left\{\mathbb{E}\left\{\sum_{i=1}^{N} U_i \mid N\right\}\right\} \\
&= \mathbb{E}\{50N\} \\
&= 500\,.
\end{aligned}
$$

For the variance, see page 112 of the text.