3 More on fair games

Definition 3.1. Let $\{X_n\}_{n=0}^{\infty}$ be a stochastic process. A *fair game* (called a *martingale* in the probability literature) with respect to $\{X_n\}_{n=0}^{\infty}$ is a stochastic process $\{M_n\}_{n=0}^{\infty}$ so that

(i) there exist a sequence of functions $\{g_n\}_{n=0}^{\infty}$ so that $M_n = g_n(X_0, \ldots, X_n)$ for all n,

(ii) $\mathbb{E}\{M_{n+1} \mid X_0, \ldots, X_n\} = M_n$.

We can rewrite condition (ii) as

$$\mathbb{E}\{M_{n+1} - M_n \mid X_0, \ldots, X_n\} = 0\,, \tag{1}$$

which is sometimes a more convenient form. Indeed, we can write

$$M_n = M_0 + \sum_{k=1}^{n} \Delta M_k \tag{2}$$

where $\Delta M_k \stackrel{\text{def}}{=} M_{k+1} - M_k$. The property (1) says that $\mathbb{E}\{\Delta M_k \mid X_0, X_1, \ldots, X_k\} = 0$ for all k. Thus (2) and (1) say that a fair game is the sum of its initial value plus increments which have expectation 0 *conditional on the past.*

We may suppose that ΔM_k is the amount we either win or lose at time k, the random variable M_0 is our initial fortune, and M_n is our fortune at time n. Equation (2) represents our current fortune as our initial fortune plus the sum of our winnings in a series of successive games. (A negative win is a loss.) The key property is that conditional on what has transpired up to time k, i.e. on the random variables $X_0, \ldots, X_k$, our expected gain on the $(k+1)$st game (that is, ΔM_k) is zero.

Example 3.2. Let $D_1, D_2, \ldots$ be a sequence of independent random variables so that $\mathbb{E}\{D_k\} = 0$ for all k. If $M_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} D_k$, then

$$\begin{aligned}\mathbb{E}\{M_{n+1} - M_n \mid D_1, \ldots, D_n\} &= \mathbb{E}\{D_{n+1} \mid D_1, \ldots, D_n\} \\ &= 0\,,\end{aligned}$$

so $\{M_n\}$ is a fair game. (It is clear that M_n is a function of $D_1, \ldots, D_n$, so condition (i) of Definition 3.1 is satisfied.)

Example 3.3. Let $\{X_n\}_{n=0}^{\infty}$ be a random walk which moves up one unit with probability p, and down one unit with probability $1 = 1 - p$, where $p \neq 1/2$. In other words, given $X_0, \ldots, X_n$,

$$\Delta X_n \overset{\text{def}}{=} X_{n+1} - X_n = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } q\,. \end{cases}$$

If $M_n \overset{\text{def}}{=} (q/p)^{X_n}$, then $\{M_n\}_{n=0}^{\infty}$ is a martingale with respect to $\{X_n\}_{n=0}^{\infty}$. Condition (i) is clear, and

$$\begin{aligned} \mathbb{E}\left\{(q/p)^{X_{n+1}} \mid X_0 = x_0, \ldots, X_n = x_n\right\} &= \mathbb{E}\left\{(q/p)^{x_n}(q/p)^{X_{n+1}-x_n} \mid X_0 = x_0, \ldots, X_n = x_n\right\} \\ &= (q/p)^{x_n}\left(p(q/p) + q(q/p)^{-1}\right) \\ &= (q/p)^{x_n}\,. \end{aligned}$$

Example 3.4. Let $\{X_n\}_{n=0}^{\infty}$ be as in the previous example. Let $\mu \overset{\text{def}}{=} p - q$, and $M_n \overset{\text{def}}{=} X_n - \mu n$. Then

$$\begin{aligned} \mathbb{E}\{M_{n+1} - M_n \mid X_0, \ldots, X_n\} &= p - q - \mu \\ &= 0\,, \end{aligned}$$

so $\{M_n\}$ is a fair game.

Definition 3.5 (Strategy). A *strategy* is a sequence of random variables $\{B_n\}_{n=0}^{\infty}$ so that $B_n = f_n(X_0, \ldots, X_{n-1})$ for some function f_n of n variables.

The point is that B_n is completely determined by what has happened strictly before time n, that is, by $X_0, \ldots, X_{n-1}$.

Suppose that $\{M_n\}_{n=0}^{\infty}$ is a martingale with respect to some stochastic process $\{X_n\}$, and $\{B_n\}$ is a strategy. We can think of B_n as the amount we are willing to stake on a game which pays out $M_{n+1} - M_n$, after observing $X_0, X_1, \ldots, X_n$. In this case, the amount that we win on this game will be

$$B_n \Delta M_n = B_n(M_{n+1} - M_n)\,.$$

The case $B_n = 0$ corresponds to a decision not to play the game, or in other words, to wager nothing. If we play n times, using the strategy $\{B_n\}$, then our total fortune will be

$$M_0 + \sum_{k=1}^{n} B_k \Delta M_{k-1} = M_0 + \sum_{k=1}^{n} B_k (M_k - M_{k-1})\,. \tag{3}$$

The point in the definition of a strategy is that we are not permitting a scheme in which we are clairvoyant (able to see into the future). For example, we would like to use the scheme that $B_n = 1$ if $M_n - M_{n-1} > 0$ and $B_n = 0$ otherwise. Then we would only bet on the games which we win, and we would only increase our fortune. We don't allow such a strategy as it presumes that we know the outcome of a game before we decide whether or not to bet.

Equation (3) motivates us to make the definition

$$(B \circ M)_n \overset{\text{def}}{=} M_0 + \sum_{k=0}^{n-1} B_k (M_k - M_{k-1})\,,$$

where $\{M_n\}$ is a fair game, and $\{B_n\}$ is a strategy.

The next result shows that if we are not allowed to look into the future, any strategy will still produce a fair game.

Theorem 3.6. *For any strategy $\{B_n\}_{n=0}^{\infty}$, the sequence of random variables $\{(B \circ M)_n\}_{n=0}^{\infty}$ is a fair game.*

Proof.

$$\begin{aligned}
\mathbb{E}\left\{(B \circ M)_{n+1} - (B \circ M)_n \mid X_0, \ldots, X_n\right\} &= \mathbb{E}\left\{B_{n+1}(M_{n+1} - M_n) \mid X_0, \ldots, X_n\right\} \\
&= B_{n+1}\mathbb{E}\left\{M_{n+1} - M_n \mid X_0, \ldots, X_n\right\} \\
&= 0\,.
\end{aligned}$$

We are allowed to move B_{n+1} outside of the conditional expectation because $X_0, \ldots, X_n$ determine the value of B_{n+1}, and so conditional on $X_0, \ldots, X_n$, the random variable B_{n+1} acts like a constant and can be moved outside the expectation. (Recall that in general if U, V are random vectors, and g is a function, $\mathbb{E}\left\{g(U)V \mid U\right\} = g(U)\mathbb{E}\left\{V \mid U\right\}$.) ■

Definition 3.7. A *stopping time* is a random variable T with values in $\{0, 1, \ldots\}$ so that the event $\{T = n\}$ is determined by the random variables $X_0, \ldots, X_n$. That is, there is some function f_n of $n + 1$ variables so that

$$\mathbf{1}\{T = n\} = f_n(X_0, \ldots, X_n)\,.$$

Example 3.8. Let $T_z = \min\{n \geq 0 \ :\ X_n = z\}$ be the first time that $\{X_n\}$ visits state z. Then

$$\mathbf{1}\{T_z = n\} = \{X_0 \neq z, X_1 \neq z, \ldots, X_{n-1} \neq z, X_n = z\}\,,$$

so T_z is a stopping time.

Example 3.9. Let

$$T = \min\{n \geq m \ :\ X_n = \max_{j \leq m} X_m\}\,.$$

Thus T is the first time after m that X_n is equal to the maximum of X_j over $j = 0, \ldots, m$. Then for $n < m$ the event $\{T = n\}$ never occurs, and so $\mathbf{1}\{T = n\} \equiv 0$, which is a trivial function of $X_0, \ldots, X_n$. For $n \geq m$,

$$\{T = n\} = \{X_m \neq \max_{j \leq m} X_j, \ldots, X_n = \max_{j \leq m} X_j\}\,,$$

and so $\mathbf{1}\{T = n\}$ is a function of $(X_0, \ldots, X_n)$. Thus T is a stopping time.

Example 3.10. Let $T = \min\{n \geq 0 \ :\ X_n = \max_j X_j\}$. Then then event $\{T = n\}$ can be writen as $\{X_n = \max_j X_j\}$, and hence depends on *all* the random variables $X_0, X_1, \ldots$ not just $(X_0, \ldots, X_n)$. Hence it is *not* a stopping time.

The next results show why stopping times are important:

Theorem 3.11. *Let T be a stopping time, and $\{M_n\}_{n=0}^{\infty}$ be a fair game. Then $\{M_{n \wedge T}\}_{n=0}^{\infty}$ is a fair game.*

Corollary 3.12. *Let $\{M_n\}_{n=0}^{\infty}$ be a fair game and T a stopping time so that $|M_{n \wedge T}| \leq K$ for all n, where K is a fixed number. Then*

$$\mathbb{E}\{M_T\} = \mathbb{E}\{M_0\}\,.$$

Proof of Theorem 3.11. Let $B_n = \mathbf{1}\{T > n\}$. Then

$$B_n = 1 - \mathbf{1}\{T \le n-1\} = 1 - \sum_{k=1}^{n-1} \mathbf{1}\{T = n\}\,,$$

and since T is a stopping time, B_n can be writen as a function of $X_0, \ldots, X_{n-1}$. Thus $\{B_n\}_{n=0}^{\infty}$ is a strategy. Check that

$$(B \circ M)_n = M_{T \wedge n} - M_0\,.$$

Thus $M_{T\wedge n} - M_0$ is a fair game. The reader should check then that $M_{T\wedge n} - M_0 + M_0 = M_{T\wedge n}$ is still a fair game. ■

Proof of Corollary 3.12. Since $\{M_{T\wedge n}\}$ is a fair game, we have

$$\mathbb{E}\left\{M_{T\wedge n}\right\} = \mathbb{E}\left\{M_0\right\}\,.$$

Thus

$$\lim_{n\to\infty} \mathbb{E}\left\{M_{T\wedge n}\right\} = \mathbb{E}\left\{M_0\right\}\,.$$

Since $M_{T\wedge n}$ is bounded, we are allowed to take a limit inside the expectation. Note that we cannot in general move a limit inside an expectation. In this case, because the random variables inside the expectation are bounded, we are permitted to do so. Thus, $\mathbb{E}\left\{M_T\right\} = \mathbb{E}\left\{M_0\right\}$. ■

Example 3.13. Let $X_0 \equiv 0$, and let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with

$$\mathbb{P}\left\{X_k = 1\right\} = \mathbb{P}\left\{X_k = -1\right\} = \frac{1}{2}\,.$$

Then $S_n = \sum_{k=0}^{n} X_k$ is a fair game. Let $B_1 \equiv 1$, and for $n > 1$, let

$$B_n = \begin{cases} 2^n & \text{if } X_1 = X_2 = \cdots = X_{n-1} = -1 \\ 0 & \text{if } X_j = 1 \text{ for some } j < n\,. \end{cases}$$

Thus, provided we have not won a single previous game, we bet 2^n, and as soon as we win, we stop playing. Then if T is the first time that we win, T is a stopping time.

$$M_n \overset{\text{def}}{=} (B \circ S)_n = \begin{cases} 0 & \text{if } n = 0\,, \\ -2^{(n-1)} & \text{if } 1 \leq n < T\,, \\ 1 & \text{if } n \geq T\,. \end{cases}$$

We have that $M_T = 1$, since we are assured that eventually $X_n = 1$ and so T is always a finite number. Thus $\mathbb{E}\left\{M_T\right\} = 1$. But $\mathbb{E}\left\{M_0\right\} = 0$, and $\{M_n\}$ is a fair game! We are playing a fair game, but have allowed that if we stop at a stopping time, we can assure ourselves a profit. This at first glance seems to contradict Corollary 3.12. But notice that the condition $|M_{T\wedge n}| < K$ is not satisfied, so we cannot apply the Corollary.

In "real life", a gambler has only finite capital, and casinos don't permit one to gamble on credit. Thus a gambler is only allowed a finite number of losses before he must quit playing (having lost all his capital.) Thus a gambler can only guarantee himself to make the random amount $M_{N\wedge T}$, where $N = \log_2 C + 1$, where C is his initial capital. (This is because after playing N games and not winning, his loss is $-2^{\log_2 C} = -C$, and he must quit.) Since $M_{n\wedge T}$ is a fair game for any fixed n, and N is a fixed number, his expected return is $\mathbb{E}\left\{M_{N\wedge T}\right\} = 0$.

4 Applications

Let X_n be a random walk, and let $\alpha(x) = \mathbb{P}_x\left\{T_0 < T_N\right\}$, where $0 \leq x \leq N$. Suppose we are in the case where $p \neq q$. We have seen before that $M_n \overset{\text{def}}{=} (q/p)^{X_n}$ is a fair game. Let $T \overset{\text{def}}{=} T_0 \wedge T_N$ be the first time the walk hits either 0 or N. Then T is a stopping time.

Since $M_{T\wedge n}$ is bounded, we can apply Corollary 3.12 to get

$$\mathbb{E}_x\left\{(q/p)^{X_T}\right\} = (q/p)^x\,.$$

We can break up the expectation above to get

$$\mathbb{E}_x\left\{(q/p)^{X_T}\right\} = \alpha(x) + (q/p)^N(1 - \alpha(x))\,.$$

Combining these two equations and solving for $\alpha(x)$ yields

$$\alpha(x) = \frac{(q/p)^x - (q/p)^N}{1 - (q/p)^N} .$$

In the case where $p = q = \frac{1}{2}$, we can apply the same argument to get that $\alpha(x) = 1 - (x/N)$.

Now consider again the unbiased random walk. Notice that

$$\begin{aligned} \mathbb{E}\left\{X_{n+1}^2 - X_n^2 \mid X_0, \ldots, X_n\right\} &= (X_n + 1)^2 \frac{1}{2} + (X_n - 1)^2 \frac{1}{2} - X_n^2 \\ &= 1 . \end{aligned}$$

Thus $M_n \overset{\text{def}}{=} S_n^2 - n$ is a fair game. By Theorem 3.11 we have that

$$\mathbb{E}_x\left\{S_{n\wedge T}^2\right\} = \mathbb{E}_x\left\{T \wedge n\right\} .$$

Now since $S_{n\wedge T}^2$ is bounded by N^2 for all n, if we take the limit as $n \to \infty$ on the left-hand side above, we can take it inside the expectation. Also, $T \wedge n$ does not decrease as n increases, so we are allowed to take the limit inside the expectation. Thus

$$\mathbb{E}_x\left\{S_T^2\right\} - x^2 = \mathbb{E}_x\left\{T\right\} .$$

Now conditioning on whether $T = T_0$ or $T = T_N$ yields

$$(1 - \alpha(x))N^2 - x^2 = \mathbb{E}_x\left\{T\right\} .$$

Hence,

$$\mathbb{E}_x\left\{T\right\} = x(N - x) .$$