Solution to Homework 3.

Problem 1.

Assume the equality holds for n. Then

$$\begin{aligned}
\mu^{n+1} &= \mu \boldsymbol{P}^n \boldsymbol{P} \\
&= \left[\frac{1}{2}\left(1+2^{-n}\right), \frac{1}{2}\left(1-2^{-n}\right)\right] \left[\begin{array}{cc} 3/4 & 1/4 \\ 1/4 & 3/4 \end{array}\right] \\
&= \left[\frac{1}{2}\left(1+2^{-n}\right)\frac{3}{4} + \frac{1}{2}\left(1-2^{-n}\right)\frac{1}{4}, \frac{1}{2}\left(1+2^{-n}\right)\frac{1}{4} + \frac{1}{2}\left(1-2^{-n}\right)\frac{3}{4}\right] \\
&= \frac{1}{8}\left[3+3\cdot 2^{-n}+1-2^{-n}, 1+2^{-n}+3-3\cdot 2^{-n}\right] \\
&= \frac{1}{8}\left[4+2\cdot 2^{-n}, 4-2\cdot 2^{-n}\right] \\
&= \frac{1}{2}\left[1+2^{-(n+1)}, 1-2^{-(n+1)}\right] \\
&= \left[\frac{1}{2}\left(1+2^{-(n+1)}\right), \frac{1}{2}\left(1-2^{-(n+1)}\right)\right].
\end{aligned}$$

Thus it holds for $n+1$. It clear holds for $n=0$. Thus it holds for all n.

We have

$$\lim_{n\to\infty} \mu^n = \left[\frac{1}{2}, \frac{1}{2}\right].$$

Problem 2. Notice that if $(Y_{n-1}, Y_n) = (0,1)$, then we must have that $X_{n-1} = 1$, as $X_{n-1} = 1$ *only if* $Y_{n-1} = 0$. This implies that $X_n = 2$. On the otherhand, we have that if $(Y_{n-1}, Y_n) = (1,1)$, then neither X_{n-1} or X_n can be 1. So, in this case, the possibilities for (X_{n-1}, X_n) are:

$$(2,2), (2,3), (3,2), (3,3).$$

The only once of these that has positive probability is $(2,3)$. Thus, if $(Y_{n-1}, Y_n) = (1,1)$, then it must be that $(X_{n-1}, X_n) = (2,3)$.

We have

$$\begin{aligned}
\mathbb{P}\left\{Y_{n+1} = 1 \mid Y_{n-1} = 1, Y_n = 1\right\} &= \mathbb{P}\left\{Y_{n+1} = 1 \mid X_{n-1} = 2, X_n = 3\right\} \\
&= 0,
\end{aligned}$$

because given that $X_n = 3$, we have with probability one that $X_{n+1} = 1$ and hence $Y_{n+1} = 0$. But

$$\begin{aligned}\mathbb{P}\{Y_{n+1} = 1 \mid Y_{n-1} = 0, Y_n = 1\} &= \mathbb{P}\{Y_{n+1} = 1 \mid X_{n-1} = 1, X_n = 2\} \\ &= 1\,,\end{aligned}$$

since X_{n+1} must be 3 when $X_n = 2$.

Thus it cannot be that

$$\mathbb{P}\{Y_{n+1} = 1 \mid Y_0 = i_0, \ldots, Y_n = i_n\} = \mathbb{P}\{Y_{n+1} = 1 \mid Y_n = i_n\}\ ,$$

as this would imply

$$\begin{aligned}0 = \mathbb{P}\{Y_{n+1} = 1 \mid Y_{n-1} = 1, Y_n = 1\} &= \mathbb{P}\{Y_{n+1} = 1 \mid Y_n = 1\} \\ &= \mathbb{P}\{Y_{n+1} = 1 \mid Y_{n-1} = 0, Y_n = 1\} = 1\end{aligned}$$

Problem 3.

Just check that

$$\mathbb{P}\{X_{2n+2} = j \mid X_2 n = i\} = (\boldsymbol{P}^2)_{i,j}\,.$$

Problem 4.

Take any state j. Since the chain is irreducible, there exist r and s so that $P^r_{i,j} > 0$ and $P^s_{j,i} > 0$. We know that

$$P^{r+s}_{j,j} \geq P^s_{j,i} P^r_{i,j} > 0\,.$$

Also, we know that

$$P^{r+s+1}_{j,i} \geq P^s_{j,i} P_{i,i} P^r_{i,j} > 0\,.$$

Thus the set of integers

$$A = \{n \ :\ P^n_{j,j} > 0\}$$

contains $r+s$ and $r+s+1$. But the g.c.d. of $r+s$ and $r+s+1$ is 1, and so the g.c.d. of A is 1. Thus state j has period 1. Since this holds for any j, the Markov chain is aperiodic.

Problem 5. The king is irreducible, as it can reach any square. It is possible to return to any position in both 2 moves and 3 moves, so it is aperiodic.

The bishop is restricted to its starting color, so it is not irreducible. It is possible to return to its position in both 2 and 3 moves, so it is aperiodic.

The knight is irreducible, but has period two. TO see that it has period two, notice that is always moves from a white position to a black position, and from a black position to a white position. Thus it can only return to its starting place after an even number of moves.