Homework M5040 - Levin

Let $\{X_n\}$ be an unbiased random walk on $\mathbb{Z}$. The goal of this assignment is to find the distribution of $T = T_{-1} = \min\{n : X_n = -1\}$. Note by symmetry that the distribution of T is the same as the distribution of T_1, the first time the walk reaches 1.

We will use generating functions. Let $A(s) = \mathbb{E}\left\{s^T\right\}$. Notice that by the definition of expectation,

$$\mathbb{E}\left\{s^T\right\} = \sum_{k=0}^{\infty} \mathbb{P}\{T = k\} s^k .$$

Suppose we find $A(s)$, and can write it as a power series $A(s) = \sum_k a_k s^k$. If a function has a power series representation (such functions are called *real analytic*), then the coefficients are unique. Thus $a_k = \mathbb{P}\{T = k\}$.

1. Show that

$$M_n \stackrel{\text{def}}{=} u^{X_n} \left(\frac{2}{u + 1/u} \right)^n$$

is a fair game.

2. Show that for u near 0, there is a constant K so that $|M_{n \wedge T}| < K$.

3. Then use the Corollary of the notes to show that

$$\mathbb{E}\left\{ \left(\frac{2}{u + 1/u} \right)^T \right\} = u . \tag{1}$$

4. Use (1) to show that

$$\mathbb{E}\left\{s^T\right\} = \frac{1}{s}\left(1 - \sqrt{1 - s^2}\right) .$$

5. Show by induction that if $f(x) = \sqrt{1 - x}$, then

$$f^{(m)}(x) = \frac{(2m-2)!}{2^{2m-1}(m-1)!}(1-x)^{-(2m-1)/2} .$$

6. Use this to show that

$$1 - \sqrt{1 - s^2} = \sum_{m=1}^{\infty} \frac{(2m-2)!}{2^{2m-1} m!(m-1)!} s^{2m} .$$

7. Conclude that

$$A(s) = \sum_{m=1}^{\infty} \frac{(2m-2)!}{2^{2m-1} m!(m-1)!} s^{2m-1} .$$

8. Now argue that

$$\mathbb{P}\{T = n\} = \begin{cases} \frac{(2m-2)!}{2^{2m-1} m!(m-1)!} & \text{if } n = 2m-1 \\ 0 & \text{if } m \text{ is even.} \end{cases}$$