Instructions

Read each problem carefully. Show all your work. Credit will only be awarded if your work is included. Each problem is worth 20 points. Bon chance!

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**Problem 1.** A couple, Andrew and Amy, are in a group of 9 people (including themselves). These 9 people sit in random order around a circular table. What is the probability that Andrew and Amy are seated next to each other?

**Solution.** There are \( \frac{9!}{9} = 8! \) ways to arrange 9 people around a circular table.

If we regard Andrew and Amy as a single unit, and the other 7 people each as a unit, we have 8 units total. The number of ways to sit these 8 units around a circular table is \( \frac{8!}{8} = 7! \). For each such arrangement, there are 2 possible orderings of Andrew and Amy. Thus the total number of ways to sit with Andrew and Amy together is \( 2 \cdot 7! \). We conclude that

\[
P(\text{Andrew and Amy are together}) = \frac{2 \cdot 7!}{8!} = \frac{1}{4}.
\]
Problem 2. Suppose an urn contains 10 each of red, blue, and white balls (for 30 balls total.) You draw 8 balls from the urn, without replacing each ball after it is drawn. Find the probability that you are missing at least one of the colors. Hint: Use the inclusion-exclusion formula.

Solution. Consider the following events:

\[ R = \{ \text{sample has no red balls} \}, \]
\[ B = \{ \text{sample has no blue balls} \}, \]
\[ W = \{ \text{sample has no white balls} \}. \]

The event we want is \( R \cup B \cup W \). We have

\[ P(R \cup B \cup W) = P(R) + P(B) + P(W) - P(RB) - P(RW) - P(BW) + P(RBW). \]

There are \( \binom{30}{8} \) ways to choose a sample of size 8. There are \( \binom{20}{8} \) ways to choose a sample which contains no red balls. There are an equal number of samples which contain no white balls, and number of samples which contain no blue balls. Thus,

\[ P(B) = P(W) = P(R) = \frac{\binom{20}{8}}{\binom{30}{8}}. \]

The number of samples which contain no red balls and no blue balls is \( \binom{10}{8} \), because then all the balls in the sample must come from the 10 white balls. Thus

\[ P(BR) = P(RW) = P(WB) = \frac{\binom{10}{8}}{\binom{30}{8}}. \]

Finally, note that \( RBW = \emptyset \), so \( P(RBW) = 0 \). We conclude that

\[ P(A \cup B \cup C) = 3 \cdot \frac{\binom{20}{8}}{\binom{30}{8}} - 3 \cdot \frac{\binom{10}{8}}{\binom{30}{8}}. \]
**Problem 3.** There are 3 coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. One of the 3 coins is selected at random and flipped twice. What is the probability that both these tosses are heads?

**Solution.** Let $C_1$ be the event that the two-headed coin is used, $C_2$ the event that the fair coin is used, and $C_3$ be the event that the biased coin is used. We have

\[
P(HH \mid C_1) = 1
\]
\[
P(HH \mid C_2) = \frac{1}{4}
\]
\[
P(HH \mid C_3) = \left(\frac{3}{4}\right)^2
\]

The last probability is arrived at because the two tosses are independent, and with the biased coin, each toss has probability 0.75 of landing heads. Next,

\[
P(HH) = P(HH \mid C_1)P(C_1) + P(HH \mid C_2)P(C_2) + P(HH \mid C_3)P(C_3)
\]
\[
= 1 \left(\frac{1}{3}\right) + \left(\frac{1}{4}\right) \left(\frac{1}{3}\right) + \left(\frac{9}{16}\right) \left(\frac{1}{3}\right)
\]
\[
= \frac{16 + 4 + 9}{48}
\]
\[
= \frac{29}{48}.
\]
**Problem 4.** Suppose that a single die is rolled 20 times. If you are told that the face 6 occurred four times, what is the conditional probability that the face 6 occurred on the first two rolls?

**Solution.** Let \( A \) be the event that the six face occurs 4 times, and let \( B \) be the event that the six face occurred on the first two rolls.

First, we compute \( P(A) \). There are 6\(^{20} \) possible outcomes, all equally likely. To calculate the number of outcomes which have 4 occurrences of the six face, we use the multiplication principle: there are \( \binom{20}{4} \) ways to specify which 4 out of 20 rolls produced a six face, and there are \( 5^{16} \) ways to specify the outcomes of the remaining 16 rolls. Thus there are \( \binom{20}{4} \cdot 5^{16} \) outcomes which have exactly 4 rolls with a six face. Consequently,

\[
P(A) = \binom{20}{4} \cdot \frac{5^{16}}{6^{20}} = \left( \frac{20}{4} \right) \cdot \left( \frac{5}{6} \right)^{16} \cdot \left( \frac{1}{6} \right)^{4}.
\]

Another way to calculate \( P(A) \) is using the independence of the rolls. There are \( \binom{20}{4} \) sequences of rolls of length 20 in which 4 are six faces, and 16 are non six-faces. The probability of each such roll is \( (1/6)^4 (5/6)^{16} \). Thus we are led to \( P(A) = \binom{20}{4} (1/6)^4 (5/6)^{16} \), the same answer we obtained above by counting.

Now we compute \( P(AB) \). The number of ways \( AB \) can occur can be determined as follows: Because we assume that the first two rolls were six faces, this leaves a remainder of 18 rolls, on which two more six faces must appear. There are \( \binom{18}{2} \cdot 5^{16} \) ways this can occur. Then

\[
P(AB) = \frac{\binom{18}{2} \cdot 5^{16}}{6^{20}} = \left( \frac{1}{6} \right)^{2} \left( \frac{18}{2} \right) \cdot \left( \frac{1}{6} \right)^{2} \cdot \left( \frac{5}{6} \right)^{16}.
\]

Another way, using independence, is

\[
P(AB) = P(A \cap \{ \text{two six faces in rolls 3 through 20} \})
= P(A)P(\{ \text{two six faces in rolls 3 through 20} \})
= \left( \frac{1}{6} \right)^{2} \left( \frac{18}{2} \right) \cdot \left( \frac{1}{6} \right)^{2} \cdot \left( \frac{5}{6} \right)^{16}.
\]

Finally,

\[
P(B \mid A) = \frac{P(AB)}{P(B)} = \frac{\binom{18}{2}}{\binom{20}{4}}.
\]
Problem 5. Suppose that a drug test can correctly identify an illegal drug user with probability 0.95, but will give a false positive with probability 0.1. Assume that 1% of the population uses illegal drugs. A person is selected at random, given the test, and tests positive. What is the conditional probability that the person is a drug user?

Solution. Let $T$ be the event the person tests positive, let $D$ be the event the person is a drug user, and let $N$ be the event the person is not a drug user. We are given the following information:

\[
P(T \mid D) = 0.95, \\
P(T \mid N) = 0.10, \\
P(D) = 0.01, \\
P(N) = 0.99.\]

Thus,

\[
P(D \mid T) = \frac{P(DT)}{P(T)} = \frac{P(T \mid D)P(D)}{P(T \mid D)P(D) + P(T \mid N)P(N)} = \frac{(0.95)(0.01)}{(0.95)(0.01) + (0.10)(0.99)} \approx 0.088.
\]