Test 3 – Math 3070 – Fall 2003

Name: ____________________________________

SHOW ALL YOUR WORK.

Problem 1. A temperature reading is normally distributed with mean μ, the actual temperature, and standard deviation σ. What would the value of σ have to be ensure that 95% of all readings are within 0.1 degree of μ?

Solution. We want to find σ so that

$$0.95 = P(|X - \mu| \leq 0.1)\ .$$

Thus, we want to solve for σ in

$$\begin{aligned} 0.95 &= P\left(\frac{|X-\mu|}{\sigma} \leq \frac{0.1}{\sigma}\right) \\ &= P\left(-\frac{0.1}{\sigma} \leq Z \leq \frac{0.1}{\sigma}\right) \\ &= \Phi\left(\frac{0.1}{\sigma}\right) - \left(1 - \Phi\left(\frac{0.1}{\sigma}\right)\right) \\ &= 2\Phi\left(\frac{0.1}{\sigma}\right) - 1 \\ 0.975 &= \Phi\left(\frac{0.1}{\sigma}\right) \end{aligned}$$

Thus

$$\begin{aligned} \frac{0.1}{\sigma} &= 1.96 \\ \sigma &= \frac{0.1}{1.96} = 0.051\,. \end{aligned}$$

□

Problem 2. Let X and Y have joint probability mass function

$$f(x,y) = \begin{cases} 2 & \text{if } 0 \le x, y \le 1 \text{ and } x + y \le 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute $P(X + Y \le t)$ for $0 \le t \le 1$.

(b) Compute the probability density function for $T = X + Y$.

Solution. Let $A_t = \{(x, y) \ : \ x + y \le t, \ \text{and } x, y \ge 0\}$.

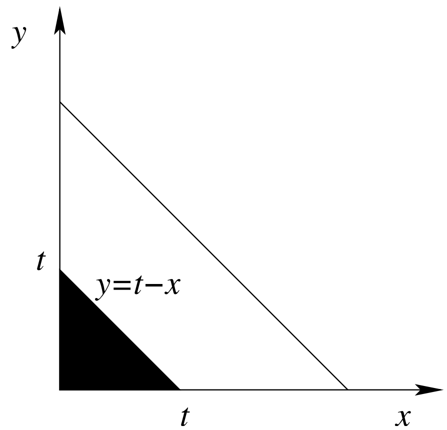

Figure 1: The shaded region is A_t

Then

$$P(X + Y \le t) = P((X, Y) \in A_t) = \iint_{A_t} 2dxdy = 2 \times \text{area}(A_t)\,.$$

We have $\text{area}(A_t) = \frac{1}{2}t^2$. Thus

$$P(X + Y \le t) = t^2\,.$$

This is the cumulative density function for T. Thus we obtain the pdf by differentiating: For $0 \le t \le 1$,

$$f(t) = \frac{d}{dt}t^2 = 2t\,,$$

and the density is 0 elsewhere. □

Problem 3. Let X and Y independent normal random variables with mean 2 and standard deviation 1. Find $E(XY)$ and $E(X+Y)$. Justify your answers.

Solution.

$$\begin{aligned} E(XY) &= E(X)E(Y) \quad \text{[by independence]} \\ &= (2)(2) \\ &= 4\,. \end{aligned}$$

$X+Y$ has a normal distribution with mean $2(2)=4$. Thus $E(X+Y)=4$. $\square$

Problem 4. Let (X, Y) be the coordinates of a point chosen randomly in the unit square of the plane. That is, the joint probability density function is given by

$$f(x,y) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Prove or disprove: X and Y are independent.

Solution. The marginal pdf of X is computed as follows: For $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 1dy = 1\,,$$

For $x < 0$ or $x > 1$, $f_X(x) = 0$. Thus

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

The marginal pdf for Y is as above, with y replacing x. Thus it is the case that

$$f_X(x) f_Y(y) = f(x,y)\,,$$

and so X and Y are independent. □

Problem 5. Suppose the times between sucessive cars on a road are independent, each with mean 10 minutes and standard deviation 3 minutes. Let $\bar{X}$ be the sample mean of a simple random sample of size 50 of these inter-arrival times. Find, approximately, the probability that $\bar{X}$ is between 9.75 and 10.25.

Solution. By the Central Limit Theorem, the distribution of $\bar{X}$ is approximately Normal with mean 10 and standard deviation $3/\sqrt{50} = 0.424$. Thus,

$$\begin{aligned} P\big(9.75 \le \bar{X} \le 10.25\big) &\approx P\left(\frac{9.75 - 10}{0.424} \le Z \le \frac{10.25 - 10}{0.424}\right) \\ &= \Phi(0.590) - \Phi(-0.590) \\ &= 0.445\,. \end{aligned}$$

□

Problem 6. Suppose that a coin is unfair, so that the probability of heads is 0.53. If I toss the coin independently 400 times, what is the probability (approximately) that the frequency of heads in my 400 tosses is more than 0.50.

Solution. Let $\bar{X}$ be the sample proportion. By the Central Limit Theorem, $\bar{X}$ is approximately normally distributed with mean 0.53 and standard deviation $\sqrt{(0.53)(0.47)/400} = 0.0250$. Thus

$$\begin{aligned} P\big(\bar{X} > 0.5\big) &= P\left(\frac{\bar{X} - 0.53}{0.0250} > \frac{0.5 - 0.53}{0.0250}\right) \\ &\approx P(Z > -1.2) \\ &= 1 - \Phi(1.2) = 0.885\,. \end{aligned}$$

□

Distribution of Test Scores

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The decimal point is 1 digit(s) to the right of the |

1 | 3
1 | 7
2 | 13
2 | 57
3 | 134
3 | 677889
4 | 001344
4 | 56899

 Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
13.00   31.50   38.00   36.08   43.75   49.00
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