

TEST 2 – M3070 – Fall 2003

SHOW ALL WORK.

NAME: _____

Problem 1. Suppose that the eggs in a carton are broken independently of each other, and denote by p the probability that a given egg is broken. A carton holds 12 eggs. How small should p be so that the probability that a carton contains no broken eggs is at least 0.85.

Solution. Let G be the event of no broken eggs in the carton, and let G_i be the event that the i th egg is not broken. Then

$$P(G) = P(G_1 \cap G_2 \cap \cdots \cap G_{12}) = P(G_1)P(G_2) \cdots P(G_{12}),$$

where the second inequality used independence. Since $P(G_i) = 1 - p$, we have

$$P(G) = (1 - p)^{12}.$$

Thus we want to solve $0.85 = (1 - p)^{12}$ for p . The solution is $p = 1 - (0.85)^{1/12} = 0.01345$. If $p \geq 0.01345$, then $P(G) \geq 0.85$. □

Problem 2. A test is developed to detect a certain disease. If it is administered to a person having the disease, it returns a correct diagnosis with probability 0.95. It will produce a false positive with probability 0.03 if given to someone without the disease. The percentage of the population with the disease is 0.001. Given that a random chosen person tests positive for the disease, what is the probability that he or she actually has the disease?

Solution. Let E be the event that the test is positive, and let D be the event that a randomly chosen person has the disease. We have

$$\begin{aligned}P(E | D) &= 0.95 \\P(E | D') &= 0.03.\end{aligned}$$

Furthermore,

$$\begin{aligned}P(D) &= 0.001 \\P(D') &= 0.999.\end{aligned}$$

We want to find $P(D | E)$:

$$\begin{aligned}P(D | E) &= \frac{P(D \cap E)}{P(E)} \\&= \frac{P(D \cap E)}{P(E \cap D) + P(E \cap D')} \\&= \frac{P(E | D)P(D)}{P(E | D)P(D) + P(E | D')P(D')} \\&= \frac{(0.95)(0.001)}{(0.95)(0.001) + (0.03)(0.999)} \\&= 0.0307.\end{aligned}$$

□

Problem 3. Suppose that X is the number of minutes I use my phone a month, and $E(X) = 65$, and $V(X) = 20$. The phone company bills me $Y = 10 + 0.25X$ dollars when I use X minutes. Find $E(Y)$ and $V(Y)$.

Solution.

$$E(Y) = E(10 + 0.25X) = 10 + 0.25E(X) = 10 + (0.25)(65) = 26.25.$$

Also,

$$V(Y) = V(10 + 0.25X) = (0.25)^2V(X) = 1.25.$$

□

Problem 4. Suppose that during my commute I go through 2 traffic lights. The first light is green with probability 0.5, and given that the first light is green, the second is green with probability 0.8. The colors of the pair of lights are independent on different days of the week. During my 5 day work week, what is the chance that on at least three days I get no red lights.

Solution. The chance of two green lights in a row is $(0.5)(0.8) = 0.4$. If X is the number of days (out of 5) with no red lights (two green lights), then X is Binomial($n = 5, p = 0.4$). So

$$P(X \geq 3) = 1 - P(X \leq 2) = 1 - B(2; 5, 0.4) = 1 - 0.683 = 0.317.$$

□

Problem 5. Suppose that traffic accidents occur at a certain intersection according to a Poisson process of rate 0.04 accidents per day.

(a) What is the expected number of accidents over a year (365 days)?

(b) What is the probability of 7 or more accidents over a period of 125 days?

Solution. If N is the number of accidents in 365 days, then N is a Poisson r.v. with parameter $\lambda = (365)(0.04) = 14.6$. Thus $E(N) = 14.6$.

If M is the number of accidents in 125 days, then M is a Poisson r.v. with parameter $\lambda = (125)(0.04) = 5$. Thus

$$P(M \geq 7) = 1 - P(M \leq 6) = 1 - F(6; 5) = 1 - 0.762.$$

□

Problem 6. Let $f(x)$ be the function defined by

$$f(x) = \begin{cases} 4x & \text{for } 0 \leq x \leq \frac{1}{2} \\ 4 - 4x & \text{for } \frac{1}{2} \leq x \leq 1 \\ 0 & \text{for } x < 0 \text{ or } x > 1. \end{cases}$$

- (a) Show that $f(x)$ is a probability density function.
- (b) If X is a random variable with probability density function $f(x)$ as above, find the cumulative distribution function of X .
- (c) Compute $P(\frac{1}{3} < X < \frac{2}{3})$.

Solution. First,

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \int_0^{1/2} 4x dx + \int_{1/2}^1 (4 - 4x) dx \\ &= 2x^2 \Big|_0^{1/2} + (4x - 2x^2) \Big|_{1/2}^1 \\ &= 2(1/2)^2 + 4(1) - 2(1)^2 - 4(1/2) + 2(1/2)^2 \\ &= 1. \end{aligned}$$

We have several cases. If $x < 0$, then $F(x) = 0$, and if $x > 1$, then $F(x) = 1$. If $0 \leq x \leq 1/2$, then

$$F(x) = \int_0^x 4y dy = 2y^2 \Big|_0^x = 2x^2.$$

If $1/2 < x \leq 1$, then

$$F(x) = \int_0^{1/2} 4y dy + \int_{1/2}^x (4 - 4y) dy = \frac{1}{2} + (4y - 2y^2) \Big|_{1/2}^x = 4x - 2x^2 - 1.$$

Thus

$$P(1/3 < X < 2/3) = F(2/3) - F(1/3) = 4(2/3) - 2(2/3)^2 - 4(1/3) + 2(1/3)^2 = \frac{2}{3}.$$

□

Problem (Extra Credit). Suppose that it is known that X has a probability mass function given by

$$p(x) = \begin{cases} \kappa x \binom{10}{x} (0.2)^x (0.8)^{10-x} & \text{if } x = 0, 1, \dots, 10 \\ 0 & \text{otherwise.} \end{cases}$$

What must the value of κ be so that $p(x)$ is a legitimate probability mass function?

Solution. We must have that $\sum_{x=0}^{10} p(x) = 1$. Hence

$$\begin{aligned} 1 &= \sum_{x=0}^{10} \kappa x \binom{10}{x} (0.2)^x (0.8)^{10-x} \\ 1 &= \kappa \sum_{x=0}^{10} x \binom{10}{x} (0.2)^x (0.8)^{10-x}. \end{aligned}$$

The sum is the same as the expectation of a Binomial random variable with parameters $n = 10, p = 0.2$. Thus it is equal to $10(0.2) = 2$. Thus we have $1 = \kappa 2$, and $\kappa = 1/2$. \square