Quiz 5 – Math 3070 – Fall 2003

Name: ______________________________________

Problem 1. Suppose that the pair of random variables (X, Y) has joint probability density function given by

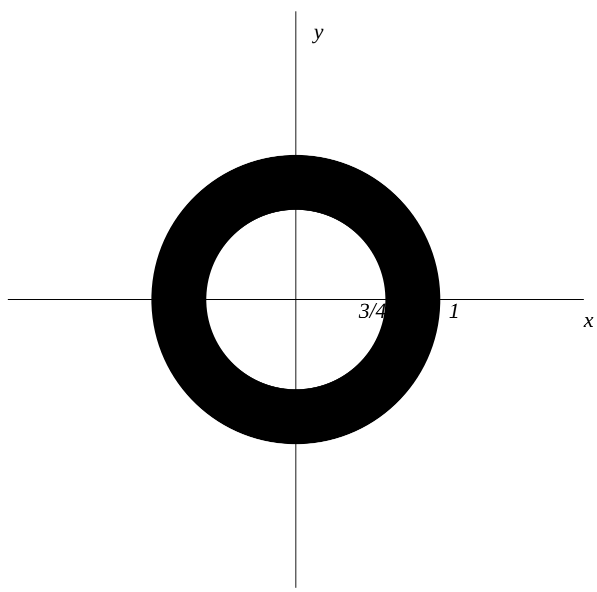
$$f(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \le 1\,, \\ 0 & \text{if } x^2 + y^2 > 1\,. \end{cases}$$

(a) Compute $P(X^2 + Y^2 > \left(\frac{2}{3}\right)^2)$.

(b) Find the marginal probability density function of X.

(c) Prove or disprove: X and Y are independent.

Solution.



Recall from our discussion that for any subset A contained in the disk of radius 1 centered at $(0, 0)$,

$$P((X,Y)\in A)=\int\int_A f(x,y)dxdy=\int\int_A \frac{1}{\pi}dxdy=\frac{\text{Area}(A)}{\pi}\,.$$

In this particular case, A is the shaded region in the figure. Thus, $\text{Area}(A)=\pi-\pi(2/3)^2$. We conclude that

$$P(X^2+Y^2>(2/3)^2)=\frac{\pi-\pi(2/3)^2}{\pi}=1-(2/3)^2\,.$$

The marginal density is obtained by integrating out y:

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)dy\,.$$

There are two cases to consider: where $x<-1$ or $x>-1$, in which case $f_X(x)=0$, and where $-1\le x\le 1$, in which case we have

$$\begin{aligned} f_X(x) &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}dy \\ &= \frac{2\sqrt{1-x^2}}{\pi}\,. \end{aligned}$$

Thus,

$$f_X(x)=\begin{cases} \frac{2\sqrt{1-x^2}}{\pi} & \text{if } |x|<1\,, \\ 0 & \text{if } |x|>1\,. \end{cases}$$

Similarly,

$$f_Y(y)=\begin{cases} \frac{2\sqrt{1-y^2}}{\pi} & \text{if } |y|<1\,, \\ 0 & \text{if } |y|>1\,. \end{cases}$$

We have for $-1\le x,y\le 1$,

$$f_X(x)f_Y(y)=\frac{4}{\pi^2}\sqrt{1-x^2}\sqrt{1-y^2}\neq f(x,y)\,,$$

so that X and Y are not independent. □