

Problem 1. Solve the following pair of equations for x and y :

$$\begin{aligned}2x - y &= 1 \\5x + 2y &= 2\end{aligned}$$

Solution. There are many ways to do this.

We can use substitution. First we solve for y in terms of x ;

$$\begin{aligned}2x - y &= 1 \\2x - 1 &= y\end{aligned}\tag{1}$$

(2)

Then we substitute in the other equation:

$$\begin{aligned}5x + 2y &= 2 \\5x + 2(2x - 1) &= 2 \\5x + 4x - 2 &= 2 \\9x &= 4 \\x &= \frac{4}{9}.\end{aligned}$$

Then we go back to Equation (1) to get y :

$$\begin{aligned}y &= 2x - 1 \\&= 2\left(\frac{4}{9}\right) - 1 \\&= -\frac{1}{9}.\end{aligned}$$

□

Alternatively, we could use matrix theory: The equations could be written as:

$$\begin{bmatrix} 2 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Since

$$\begin{bmatrix} 2 & -1 \\ 5 & 2 \end{bmatrix}^{-1} = \frac{1}{9} \begin{bmatrix} 2 & 1 \\ -5 & 2 \end{bmatrix}$$

it follows that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 2 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

So $x = \frac{4}{9}$ and $y = -\frac{1}{9}$

Problem 2. Suppose that there are 40 students in a class made up of sophomores, and juniors, and seniors. There are three times as many juniors as sophomores. The number of seniors equals the number of sophomores and juniors combined. How many sophomores, and juniors, and seniors are there?

Solution. Let x, y, z denotes the numbers of sophomores, and juniors, and seniors respectively. We have

$$\begin{aligned} -2x + y &= 0 \\ x + y - z &= 0 \\ x + y + z &= 40. \end{aligned}$$

Let us get into reduced echelon form the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 40 \\ 1 & 1 & -1 & 0 \\ -3 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 & 40 \\ 1 & 1 & -1 & 0 \\ -3 & 1 & 0 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 1 & 1 & 40 \\ 0 & 0 & -2 & -40 \\ 0 & 4 & 3 & 120 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 40 \\ 0 & 4 & 3 & 120 \\ 0 & 0 & 1 & 20 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 1 & 0 & 20 \\ 0 & 4 & 0 & 60 \\ 0 & 0 & 1 & 20 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 20 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & 20 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & 20 \end{bmatrix} \end{aligned}$$

Thus $x = 5, y = 15, z = 20$.

□

Problem 3. Let

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}$$

Find AB .

Solution.

$$\begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} (2)(1) + (-3)(-4) & (2)(2) + (-3)(3) \\ (1)(1) + (0)(-4) & (1)(2) + (0)(3) \end{bmatrix} = \begin{bmatrix} 14 & -5 \\ 1 & 2 \end{bmatrix}$$

□

Problem 4. If A is the matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

then it has an inverse

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

Use this to solve the system of equations

$$x - y + z = 2$$

$$x \quad - z = 4$$

$$x + y + z = 3$$

Solution. We have

$$A^{-1} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} (1)(2) + (2)(4) + (1)(3) \\ (-2)(2) + (0)(4) + (2)(3) \\ (1)(2) + (-2)(4) + (1)(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 13 \\ 2 \\ -3 \end{bmatrix}$$

Thus $x = \frac{13}{4}, y = \frac{1}{2}, z = -\frac{3}{4}$.

□

Problem 5. Find the inverse of the matrix

$$\begin{bmatrix} 5 & 2 \\ 4 & 1 \end{bmatrix}$$

Solution. We have

$$\begin{bmatrix} 5 & 2 \\ 4 & 1 \end{bmatrix}^{-1} = \frac{1}{-3} \begin{bmatrix} 1 & -2 \\ -4 & 5 \end{bmatrix}$$

□

Problem 6. Find the determinant of the following matrix

$$\begin{bmatrix} 1 & 5 & 50 \\ 2 & 1 & 100 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution. Expanding the determinant along the bottom row, we have

$$\det = 0 \det \begin{bmatrix} 5 & 50 \\ 1 & 100 \end{bmatrix} - 0 \det \begin{bmatrix} 1 & 50 \\ 2 & 100 \end{bmatrix} + 3 \det \begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix} = 3(1 - 10) = -27.$$

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