Math 1050-1

Spring 2004

Test 3

NAME ______________________________

Show all your work!

Problem 1. Graph the following function:

$$f(x) = \frac{x+4}{x^2+3x-18}\,.$$

Label all features of the graph.

Solution. The denomenator factors as

$$x^2 + 3x - 18 = (x+6)(x-3)\,,$$

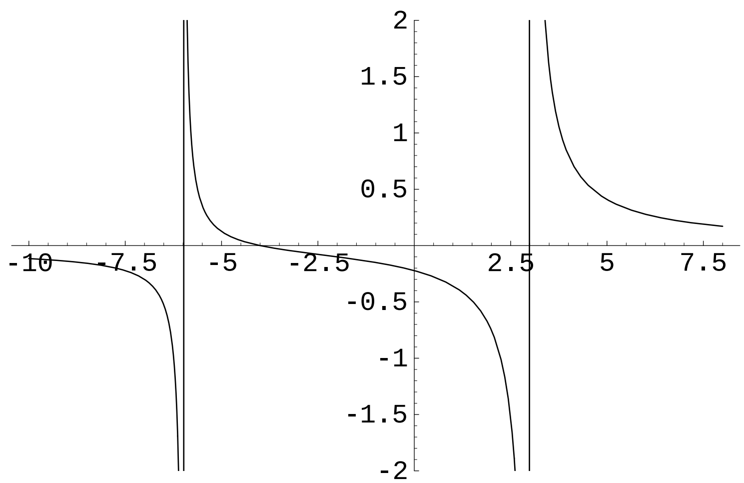
and so is zero at $x = -6$ and $x = 3$. Thus f has vertical asymptotes at $x = -6$ and $x = 3$.

The numerator is 0 at $x = -4$, and thus the function has a single zero at $x = -4$.

Since the numerator has degree one, which is smaller than the degree of the denominator (two), there is a horizontal asymptote at $y = 0$.

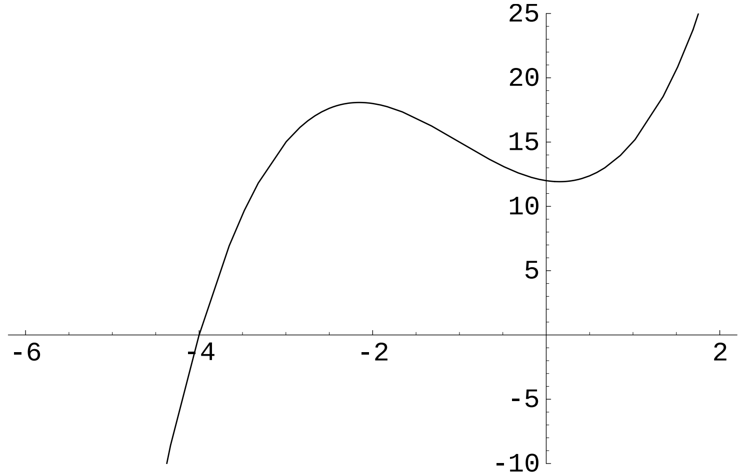
Now we only need to check the value of the function at a point between each 0 and vertical asymptote:

interval	test x in interval	$f(x)$	$\pm$
$(-\infty, -6)$	-7	$-\frac{3}{10}$	$-$
$(-6, -4)$	-5	$\frac{1}{8}$	$+$
$(-4, 3)$	0	$\frac{-2}{9}$	$-$
$(3, \infty)$	4	$\frac{4}{5}$	$+$



□

Problem 2. Below is a graph of $f(x) = x^3 + 3x^2 - x + 12$. Find all its (possibly complex) roots.



Solution. Because f has a zero at $x = -4$ (seen from the graph), $x + 4$ is a factor. Long division gives

$$
\begin{array}{r|lllll}
 & & x^2 - x + 3 & & \\
\hline
x+4 & x^3 & +3x^2 & -x & +12 \\
 & x^3 & +4x^2 & & \\
 & & -x^2 & -x & \\
 & & -x^2 & -4x & \\
 & & & 3x & +12 \\
 & & & 3x & +12 \\
 & & & & 0
\end{array}
$$

That is,

$$f(x) = (x + 4)(x^2 - x + 3)\,.$$

To find the roots of f, we must find the roots of the quadratic. The quadratic formula gives

$$
\begin{aligned}
x &= \frac{1 \pm \sqrt{1 - 4(1)(3)}}{2} \\
&= \frac{1 \pm \sqrt{-11}}{2} \\
&= \frac{1}{2} \pm i\frac{\sqrt{11}}{2}\,.
\end{aligned}
$$

Thus the three roots of f are

$$-4, \quad \frac{1}{2} + i\frac{\sqrt{11}}{2}, \quad \frac{1}{2} - i\frac{\sqrt{11}}{2}\,.$$

$\square$

Problem 3. Compute each of the following

(i) $\log_\pi(\pi^\pi)$

(ii) $\sqrt{3}^{\log_{\sqrt{3}}(3^3)}$

(iii) $\log_{123}(e) + \log_{123}\left(\frac{123^{3/2}}{e}\right)$.

Solution.

$$\log_\pi(\pi^\pi) = \pi \log_\pi(\pi) = \pi\,.$$

$$\sqrt{3}^{\log_{\sqrt{3}}(3^3)} = 3^3 = 9\,.$$

$$\log_{123}(e) + \log_{123}\left(\frac{123^{3/2}}{e}\right) = \log_{123}\left(e\frac{123^{3/2}}{e}\right) = \log_{123} 123^{3/2} = \frac{3}{2}\log_{123} 123 = \frac{3}{2}$$

□

Problem 4. Suppose that a stock price is cut in half every 10 days. It starts out at 45 dollars/share. Let $f(t)$ be the value of the stock after t days.

(i) Find $f(t)$.

(ii) After how many days will the stock be worth 1 dollar/share?

Solution.

$$f(t) = 45(1/2)^{t/10}\,,$$

either by remembering the formula, or by reasoning as follows:

$f(t)$ must be an exponential function, so $f(t) = Ba^t$ for some B and some a. We know

$$f(t+10) = \frac{1}{2}f(t)\,,$$

so plugging in the formula $f(t) = Ba^t$ gives

$$\frac{1}{2} = \frac{Ba^{t+10}}{Ba^t} = a^{10}\,.$$

Solving for a gives $a = \left(\frac{1}{2}\right)^{1/10}$. We know that B is the initial value, so $B = 45$.

For the second part, we need to solve the equation $f(t) = 1$ for t.

$$\begin{aligned} 1 &= (45)2^{-t/10} \\ \frac{1}{45} &= 2^{-t/10} \\ \log_2\left(\frac{1}{45}\right) &= -\frac{t}{10} \\ -10\log_2\left(\frac{1}{45}\right) &= t\,. \end{aligned}$$

Thus after $t = -10\log_2\left(\frac{1}{45}\right)$ days, the stock is worth 1 dollar/share. To compute the $\log_2$, use the change of base formula:

$$\log_2\left(\frac{1}{45}\right) = \frac{\ln\left(\frac{1}{45}\right)}{\ln 2}\,.$$

The answer simplifies to 54.92. □

Problem 5. Solve the following equation for x:

$$\log_4(3x^3 - 5) = 3\,.$$

Solution.

$$\begin{aligned} \log_4(3x^3 - 5) &= 3 \\ 4^{\log_4(3x^3-5)} &= 4^3 \\ 3x^3 - 5 &= 64 \\ 3x^3 &= 69 \\ x^3 &= 23 \\ x &= 23^{1/3}\,. \end{aligned}$$

□

Problem 6. Solve for x:

$$e^{\sqrt{x}-2} = \sqrt{3}\,.$$

Solution.

$$\begin{aligned}
e^{\sqrt{x}-2} &= \sqrt{3} \\
\ln\left(e^{\sqrt{x}-2}\right) &= \ln(\sqrt{3}) \\
\sqrt{x} - 2 &= \ln(\sqrt{3}) \\
\sqrt{x} &= 2 + \ln(\sqrt{3}) \\
x &= \left(2 + \ln(\sqrt{3})\right)^2 \,.
\end{aligned}$$

□

Problem 7. Suppose a function f of x triples every positive change of 5 units in x, and has initial value 6. What is $f(x)$?

Solution. One either remembers the formula, and gets

$$f(x) = (6)3^{x/5},$$

or reasons as follows:

The function has the form $f(x) = Ba^x$, since it is of exponential form. We know B must be the inital value 6. We have $f(x+5) = 3f(x)$, so plugging in the formula $f(x) = Ba^x$ gives

$$\begin{aligned} Ba^{x+5} &= 3Ba^x \\ a^5 &= 3 \\ a &= 3^{1/5}. \end{aligned}$$

Thus $f(x) = (6)(3^{1/5})^x = (6)3^{x/5}$. □

Problem 8. Suppose you deposit 250 dollars in a bank account which compounds interest continuously. After 140 days, you have 270 dollars. What is the nominal interest rate?

Solution. We have $f(t) = Pe^{rt}$, where P is the initial investment, and r is the nominal interest rate. $P = 250$ here.

$$\frac{270}{250} = \frac{f(140)}{250} = \frac{250e^{r140}}{250} = e^{140r}\,.$$

This gives the equation $27/25 = e^{140r}$. Solving for r gives:

$$\begin{aligned} e^{140r} &= \frac{27}{25} \\ 140r &= \ln\left(\frac{27}{25}\right) \\ r &= \frac{1}{140}\ln\left(\frac{27}{25}\right) \\ r &= 0.000549722\,. \end{aligned}$$

This gives the interest rate in units of days. □

Problem 9. Suppose a population grows at rate 100 individuals per day from an initial size of 20000. The population size approaches a stable maximum size of 235000. Find the function giving population size as a function of time (measured in days).

Solution. Because the population size stablizes, we use a logistic function:

$$f(x) = \frac{a}{1 + be^{-rt}} .$$

Here $r = 100$, and $a = 235000$. The only thing left is to determine b. We use the equation $f(0) = 20000$ to find b:

$$\begin{aligned} 20000 &= f(0) \\ &= \frac{235000}{1+b} \\ 20000(1+b) &= 235000 \\ 1+b &= \frac{235000}{20000} \\ b &= \frac{235}{20} - 1 \\ b &= \frac{43}{4} . \end{aligned}$$

Thus

$$f(t) = \frac{235000}{1 + \frac{43}{4}e^{-100t}} .$$

□

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The decimal point is 1 digit(s) to the right of the |

0 | 69
1 | 00
1 | 5778
2 | 022
2 | 679
3 | 0144
3 | 6778999
4 | 011223444
4 | 688999
5 | 11244
5 | 778
6 | 000011223444444
6 | 55577777889
7 | 1222334
7 | 567778888899
8 | 0011112222333444
8 | 555556667778889
9 | 00000

 Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
 6.00   44.00   65.00   61.33   81.00   90.00
```