**Generation**

- A full subcategory $\mathcal{S}$ is **thick** if it is closed under:
  - direct summands,
  - suspensions,
  - cones.
- $\text{thick}_T(X) = \text{smallest thick subcategory of } T$ containing $X$.
- If $Y \in \text{thick}_T(X)$, then $X$ *generates* $Y$.

Write $X \xrightarrow{\sim} Y$. This is constructive:

$X \rightsquigarrow \xrightarrow{\text{finite direct sum}} \xrightarrow{\text{suspensions}} \xrightarrow{\text{cones}} Y$.

- $\text{level}_T^L(Y) = \text{minimal number of cones}$

**Examples**

1. $\text{Perf}(R) = \text{perfect complexes}$
   Then $\text{thick}_T(R) = \text{Perf}(R)$.

2. $M$ a f.g. $R$-module then $\text{level}_T^L(M) = \text{pd}_R(M) + 1$.

3. $\text{thick}_T(k) = \text{complexes with bounded finite length homology}$

**Theorem**

Let $R$ be a $k$-algebra essentially of finite type over $k$. Then TFAE

1. $R$ is a locally complete intersection,
2. $R$ is proxy small in $D(R^e)$ where $R^e = R \otimes_k R$.

**Proof sketch:**

1. $\implies$ (2): $R$ c.i. $\implies$ $R^e$ c.i. $\implies$ $R$ is proxy small in $D(R^e)$.
2. $\implies$ (1): For $X \in D_f(R)$ one has

$$D(R^e) \xrightarrow{-\otimes_k X} D(R)$$

$$R \xrightarrow{R^e} W \implies R \otimes_k^L X = X \xrightarrow{R^e} W \otimes_k^L X \in \text{thick}_R(\text{Add}(R))$$

So $W \otimes_k^L X$ is small. It remains to show $\text{Supp}_R(X) = \text{Supp}_R(W \otimes_k^L X)$. Idea:

Support $\leftrightarrow$ Localizing subcategory.

**Characterizations**

**TFAE**

1. $X$ regular.
2. every object in $D_f(R)$ is small.

**TFAE**

1. $R$ a locally complete intersection.
2. every object in $D_f(R)$ is proxy small.

**Proxy small objects**

A complex $X \in D(R)$ is **small** if

$$R \xrightarrow{\sim} X$$

A complex $X \in D(R)$ is **proxy small** if there exists a small object $W$, such that

$$X \xrightarrow{\sim} W \text{ and } \text{Supp}_R(W) = \text{Supp}_R(X).$$

**Examples**

1. small $\implies$ proxy small.
2. $(R, m, k)$ local, and $K^R = \text{Koszul complex}$. Then
   (a) $k \xrightarrow{\sim} K^R$
   (b) $\text{Supp}_R(K^R) = \{m\} = \text{Supp}_R(k)$.

**Lemma**

For $X \in D_f(R)$ TFAE

1. $X$ is proxy small.
2. $X_p$ is proxy small $\forall p \in \text{Spec}(R)$.

**References**

