Restriction of the Oscillator Representation to Dual Pairs of Type I

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Motivation

Classical problem: Understanding the restriction of a representation Π of G to a subgroup H. We focus on $(\mathfrak{g}, K)$-modules.

- Hard: Analyze $\mathrm{Hom}_{(\mathfrak{h},K_H)}(\Pi,\pi)$ for π an $(\mathfrak{h}, K_H)$-module
- Related: Use derived functors, calculate $\mathrm{Ext}^n_{(\mathfrak{h},K_H)}(\Pi,\pi)$
- If the restriction is projective: $\mathrm{Ext}^n_{(\mathfrak{h},K_H)}(\Pi,\pi) = 0\ \forall n > 0$.

Main tool: Dual pairs and duality correspondence.

- **Dual pair:** (G, G') in $Sp(2N,\mathbb{R})$ is a dual pair if G and G' are centralizers of each other inside $Sp(2N,\mathbb{R})$.
- **Duality correspondence:** List of the pairs of representations of G and G' appearing in the oscillator representation.

Idea: We can use G to compute the restriction to G'.

Goal: Give a criterion for a dual pair (G, G') in $Sp(V)$ so that the restriction of the oscillator representation of $Sp(V)$ to G' is a projective $(\mathfrak{g}', K')$-module.

Oscillator Representation

- Also called Segal–Shale–Weil representation, Weil representation, harmonic representation, or metaplectic representation, among many other names.
- Start from the oscillator representation $\widetilde{\omega}$ of the metaplectic group $\widetilde{Sp}(2N,\mathbb{R})$, a double cover of the symplectic group.
- Fock model of the oscillator representation: Realization of $\widetilde{\omega}$ as an $(\mathfrak{sp}(2N,\mathbb{C}), \widetilde{U}(N))$-module. Still called oscillator representation, but denoted by ω.
- K-types of the oscillator representation: Indexed by the weights $(m+\frac{1}{2},\dots,\frac{1}{2})$ of $\widetilde{U}(N)$, where $m = 0,1,2,\dots$.
- Two irreducible summands: One with the K-types for m even, one with the K-types for m odd.

Set-up

Groups and Lie algebras:

- G, G' real Lie subgroups of $Sp(V)$
  - (G, G') dual pair
  - G' the smaller member
  - $\mathfrak{g}, \mathfrak{g}'$ complexified Lie algebras
  - $\mathfrak{t}$ Cartan subalgebra of $\mathfrak{g}$, with corresponding root system Δ
- K, K' maximal compact in G, G'
  - $\mathfrak{k}, \mathfrak{k}'$ complexified Lie algebras
  - Compact roots Δ_c: roots from $\mathfrak{k}$
  - Non-compact roots $\Delta_n = \Delta - \Delta_c$
- Cartan decompositions
  - $\mathfrak{g} = \underbrace{\mathfrak{k} + \mathfrak{p}_+}_{\mathfrak{q}} + \mathfrak{p}_-$
  - $\mathfrak{g}' = \underbrace{\mathfrak{k}' + \mathfrak{p}'_+}_{\mathfrak{q}'} + \mathfrak{p}'_-$
- M' centralizer of K in $Sp(V)$
  - $\big((K, M'), (G, G')\big)$ seesaw pair

$$\begin{array}{ccc} G & & M' \\ \cup & \times & \cup \\ K & & G' \end{array}$$

  - $\mathfrak{m}'$ complexified Lie algebra

Highest weight modules:

- F_λ irreducible $\mathfrak{k}$-module with highest weight λ
- E_λ irreducible $\mathfrak{g}$-module with highest weight λ
- $N(\lambda)$ generalized Verma module $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{q})} F_\lambda$, a $\mathfrak{U}(\mathfrak{g})$-module

If $N(\lambda)$ is irreducible, then it is equal to E_λ.

$(\mathfrak{g}, K)$-modules

Complex vector space V with an action of $\mathfrak{g}$ and an action of K such that

1. for all $v \in V, k \in K, X \in \mathfrak{g}$,
$$k \cdot (X \cdot v) = (Ad(k)X) \cdot (k \cdot v),$$
2. V is K-finite, i.e., for every $v \in V$, the space generated by $K \cdot v$ is a finite-dimensional vector space,
3. for all $v \in V, Y \in \mathfrak{k}$,
$$\left(\frac{d}{dt}\exp(tY)\cdot v\right)\Big|_{t=0} = Y \cdot v.$$

What about projectivity?

Any $(\mathfrak{k}, K)$-module V is K-finite so projective. By Frobenius reciprocity, $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{k})} V$ is a projective $(\mathfrak{g}, K)$-module.

Duality

For a pair (G, G'): Decompose the oscillator representation ω under the action of G to get

$$\omega = \bigoplus_{\sigma\in\mathrm{Irr}(G)} \underbrace{(\mathrm{Hom}_G(\sigma,\omega)}_{\theta(\sigma)} \otimes \sigma).$$

- This is a complete decomposition of ω when G is compact.
- When G is not compact, we use a seesaw dual pair containing (G, G') and the maximal compact subgroup K of G.
  $\hookrightarrow$ see case 1.

Duality correspondence:

The duality correspondence gives an explicit description of all $\theta(\sigma)$ appearing in ω.

Case 1: $G' \neq M'$

- G not compact, so use K-action to decompose
$$\omega = \bigoplus_{\sigma\in\mathrm{Irr}(K)} (\sigma \otimes E_\tau),$$
with E_τ an irreducible representation of M' with highest weight τ.
- Compute a condition so that $N(\tau)$ is irreducible, hence $N(\tau) = E_\tau$.
- Compute the restriction from M' to G' to get
$$\omega \cong \bigoplus_{\sigma\in\mathrm{Irr}(K)} \left(\sigma \otimes \left(\mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{k}')} (F_\tau\,|_{\mathfrak{k}'})\right)\right).$$
- Each summand is a projective $(\mathfrak{g}', K')$-module, under the condition above.

Case 2: $(K, M') = (K_1, G') \oplus (K_2, G')$

- K_1, K_2 of same type: Decompose $\omega = \omega_1 \otimes \omega_2^*$, with ω_1 highest K_1-module module, ω_2^* lowest K_2-module module.
- Decompose each piece: $\omega_1 = \bigoplus_{\sigma\in\mathrm{Irr}(K_1)} (\sigma \otimes E_\tau)$ and $\omega_2 = \bigoplus_{\sigma\in\mathrm{Irr}(K_2)} (\widetilde{\sigma} \otimes E_{\widetilde{\tau}})$.
- Compute conditions so that $N(\tau)$ and $N(\widetilde{\tau})$ are irreducible. It implies that $N(\tau) = E_\tau, N(\widetilde{\tau}) = E_{\widetilde{\tau}}$.
- Compute the tensor product
$$\omega = \omega_1 \otimes \omega_2^* = \bigoplus_{\substack{\sigma\in\mathrm{Irr}(K_1),\\ \widetilde{\sigma}\in\mathrm{Irr}(K_2)}} \left((\sigma\otimes\widetilde{\sigma}) \otimes \left(\mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{k}')} (F_\tau \otimes F_{\widetilde{\tau}})\right)\right).$$
- Under both conditions above, each summand is a projective $(\mathfrak{g}', K')$-module.

Bound calculation

- Irreducibility of $N(\tau)$ determines the condition on (G, G').
- As a bound for the irreducibilty, it cannot be improved.
- As a bound for projectivity, it might be improved with other methods.

Criterion: Assume that for any non-compact positive root α with
$$\frac{2\langle\lambda+\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}_{>0},$$
there is a non-compact root γ such that
$$\frac{2\langle\lambda+\rho,\gamma\rangle}{\langle\gamma,\gamma\rangle} = 0$$
and
$$s_\alpha(\gamma) \text{ is compact.}$$
Then $N(\lambda)$ is irreducible.

Results

Theorem. *Let G' be the smaller member of a dual pair (G, G') in a symplectic group $Sp(V)$. Then the restriction of the Fock model of the oscillator representation ω of $\widetilde{Sp}(V)$ to G' is a projective $(\mathfrak{g}', K')$-module under the condition (*) listed in the table below. The stable range is given for comparison.*

	(G, G')	(K, M')	(*)	*Stable range*
(i)	$(Sp(2n,\mathbb{R}), O(p,q))$	$(U(n), U(p,q))$	$n \geqslant p+q-1$	$n \geqslant p+q$
(ii)	$(O(p,q), Sp(2n,\mathbb{R}))$	$(O(p), Sp(2n,\mathbb{R})) \oplus (O(q), Sp(2n,\mathbb{R}))$	$p,q \geqslant 2n-1$	$p,q \geqslant 2n$
(iii)	$(O^*(2n), Sp(p,q))$	$(U(2n), U(2p,2q))$	$n \geqslant 2(p+q)-1$	$n \geqslant 2(p+q)$
(iv)	$(Sp(p,q), O^*(2n))$	$(Sp(p), O^*(2n)) \oplus (Sp(q), O^*(2n))$	$p,q \geqslant n-1$	$p,q \geqslant n$
(v)	$(Sp(2n,\mathbb{C}), O(p,\mathbb{C}))$	$(Sp(n), O^*(2p))$	$n \geqslant p-1$	$n \geqslant p$
(vi)	$(O(p,\mathbb{C}), Sp(2n,\mathbb{C}))$	$(O(p), Sp(4n,\mathbb{R}))$	$p \geqslant 4n-1$	$p \geqslant 4n$
(vii)	$(U(r,s), U(p,q))$	$(U(r), U(p,q)) \oplus (U(s), U(p,q))$	$r,s \geqslant p+q-1$	$r,s \geqslant p+q$

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