**RESTRICTION OF THE OSCILLATOR REPRESENTATION TO DUAL PAIRS OF TYPE I**

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**Motivation**

Classical problem: Understanding the restriction of a representation of a Lie group to a subgroup. We focus on $(g, K)$-modules.

- Hard: Analyze $\text{Hom}_{B_k}(\varPi, \pi)$ for $\pi$ an $(k, K)$-module.
- Related: Use derived functors, calculate $\text{Ext}^p_{B_k}(\varPi, \pi)$.
- If the restriction is projective: $\text{Ext}^p_{B_k}(\varPi, \pi) = 0 \forall n > 0$.

Main tool: Dual pairs and duality correspondence.

- **Dual pair**: $(G, G')$ in $Sp(2N, R)$ is a dual pair if $G$ and $G'$ are centralizers of each other inside $Sp(2N, R)$.
- **Duality correspondence**: List of the pairs of representations of $G$ and $G'$ appearing in the oscillator representation. Idea: We can use $G$ to compute the restriction to $G'$.

**Goal**: Give a criterion for a dual pair $(G, G')$ in $Sp(V)$ so that the restriction of the oscillator representation of $Sp(V)$ to $G'$ is a projective $(g', K')$-module.

**Oscillator representation**

- Also called Segal-Shale-Weil representation. Weil representation, harmonic representation, or metaplectic representation, among many other names.
- Start from the oscillator representation $\mathcal{O}$ of the metaplectic group $Sp(2N, R)$, a double cover of the symplectic group.
- Fock model of the oscillator representation: Realization of $\mathcal{O}$ as an $(sp(2N, C), U(\mathcal{N}))$-module. Still called oscillator representation, but denoted by $\omega$.
- $K$-types of the oscillator representation: Indexed by the weights $(m + \frac{1}{2}, \ldots, \frac{1}{2})$ of $U(\mathcal{N})$, where $m = 0, 1, 2, \ldots$.
- Two irreducible summands: One with the $K$-types for $m$ even, one with the $K$-types for $m$ odd.

**Set-up**

**Groups and Lie algebras**

- $(G, K)$ real Lie subgroups of $Sp(V)$
- $(g, K)$ dual pair
- $G'$ the smaller member
- $g'$ complexified Lie algebras
- $\varPi$ Cartan subalgebra of $g$, with corresponding root system $\Delta$
- $K, K'$ maximal compact in $G, G'$
- $\mathfrak{t}$, $\mathfrak{t}'$ complexified Lie algebras
- Compact roots $\Delta_c$; roots from $\mathfrak{t}$
- Non-compact roots $\Delta_n = \Delta - \Delta_c$

**Cartan decompositions**

- $g = \mathfrak{t} + \mathfrak{p} + \mathfrak{p}'$
- $g' = \mathfrak{t}' + \mathfrak{p}' + \mathfrak{p}''$

**$M'$ centralizer of $K$ in $Sp(V)$**

- $\{(K, M'), (G, G')\}$ seesaw pair

**$(g, K)$-modules**

Complex vector space $V$ with an action of $g$ and an action of $K$ such that

1. for all $v \in V, k \in K, X \in g$,
   
   $k \cdot (X \cdot v) = (Ad_k(X)) \cdot (k \cdot v)$.
2. $V$ is $K'$-finite, i.e., for every $v \in V$, the space generated by $K' \cdot v$ is a finite-dimensional vector space.
3. for all $v \in V, Y \in \mathfrak{t}$,

   $\left( \frac{d}{dt} \exp(tv) \cdot v \right) |_{t=0} = Y \cdot v$.

What about projectivity?

Any $(K, M')$-module $V$ is $K'$-finite so projective. For Frobenius reciprocity, $\text{Hom}(g, \mathfrak{t}) \otimes_{\mathfrak{t}} V$ is a projective $(g, K)$-module.

**Highest weight modules**

- $F_k$, irreducible $\mathfrak{t}$-module with highest weight $k$
- $E_k$, irreducible $g'$-module with highest weight $k$
- $\mathcal{N}(\lambda)$ generalized Verma module $\varPi(\lambda) \otimes_{\mathfrak{t}} E_k$, a $\varPi(\lambda)$-module.

If $N(\lambda)$ is irreducible, then it is equal to $E_k$.

**Duality**

For a pair $(G, G')$: Decompose the oscillator representation $\omega$ under the action of $G$ to get

$$
\omega = \bigoplus_{\sigma \in \varPi(G)} \left( \sigma \otimes E_\sigma \right).
$$

- This is a complete decomposition of $\omega$ when $G$ is compact.
- When $G$ is not compact, we use a seesaw dual pair containing $(G, G')$ and the maximal compact subgroup $K$ of $G$.

**Duality correspondence**

The duality correspondence gives an explicit description of all $\otimes (\sigma, \lambda)$ appearing in $\omega$.

<table>
<thead>
<tr>
<th>$(G, G')$</th>
<th>$(K, M')$</th>
<th>$(\lambda)$</th>
<th>Stable range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(Sp(2N, R), O(p, q))$</td>
<td>$(U(n), U(p, q))$</td>
<td></td>
<td>$n \geq p + q$</td>
</tr>
<tr>
<td>$(O(p, q), Sp(2N, R))$</td>
<td>$(O(p, q), Sp(2N, R))$</td>
<td></td>
<td>$n \geq 2(n - 1)$</td>
</tr>
<tr>
<td>$(O^*(2n), Sp(2N, R))$</td>
<td>$(U(2n), U(2p, 2q))$</td>
<td></td>
<td>$n \geq 2(p + q)$</td>
</tr>
<tr>
<td>$(Sp(p, q), O^*(2n))$</td>
<td>$(Sp(p, O^<em>(2n)) \otimes (Sp(q), O^</em>(2n))$</td>
<td></td>
<td>$n \geq n - 1$</td>
</tr>
<tr>
<td>$(Sp(2n, C), O(p, C))$</td>
<td>$(Sp(p, O^*(2n))$</td>
<td></td>
<td>$n \geq p - 1$</td>
</tr>
<tr>
<td>$(O(p, C), Sp(2n, C))$</td>
<td>$(O(p, Sp(4n, R))$</td>
<td></td>
<td>$n \geq 4n$</td>
</tr>
<tr>
<td>$(U(r, s), O(p, q))$</td>
<td>$(U(\mathcal{U}, U(p, q)) \otimes (U(s), U(p, q))$</td>
<td></td>
<td>$r, s \geq p + q$</td>
</tr>
</tbody>
</table>

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**Bibliography**

