
Dr. Enka Lakuriqi
University of Utah
Department of Mathematics
Final Exam

May 5, 2009, 1.00-3.00pm, JWB 335
Calculus II (MATH 1220-004)
Name: SOLUTIONS

Question	Score	Question	Score
1		9	
2		10	
3		11	
4		12	
5		13	
6		14	
7		15	
8		Bonus	
		TOTAL	

Instructions

1. The exam lasts 120 minutes. Do not open this booklet, nor start before you are given permission to.
2. There are 15 multiple choice questions worth 2 points + an open bonus question worth 4 points.
3. Mark your answer *unambiguously* by circling it!
4. Show all relevant work. The grading scheme that I will follow is this: *2 points* for a correct and supported answer, *1 point* for a solution that could lead to the correct answer but has minor mistakes, and *0 points* for wrong, unjustified (i.e., guessed) or blank answer.
5. You are permitted to use a (double sided) handwritten note sheet. No calculators, other class notes or books can be used.
6. You can use the last sheet of this booklet for lengthy computations. Note that I will not refer to this page while grading.
7. Please make sure that any electronic device that you possess is turned off before you enter the exam room.
8. Relax and good luck!

1. Let

$$f(x) = \ln(3x^2 + 1 + e^{-x}).$$

Find $f'(0)$.

(A) -1

(B) 0

(C) 1/2

(D) -1/2

(E) e^{-1}

(F) \sqrt{e}

$$\begin{aligned} f'(x) &= \frac{1}{3x^2 + 1 + e^{-x}} \cdot (3x^2 + 1 + e^{-x})' \\ &= \frac{6x - e^{-x}}{3x^2 + 1 + e^{-x}} \end{aligned}$$

$$f'(0) = \frac{6 \cdot 0 - e^{-0}}{3 \cdot 0^2 + 1 + e^{-0}} = \frac{-1}{2}$$

$$\boxed{f'(0) = -\frac{1}{2}}$$

2. Find the value of the following limit.

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3}. \quad [\text{Indeterminate form: } \frac{0}{0}]$$

(A) 0

(B) 1

(C) 1/2

(D) 1/3

(E) 1/6

(F) ∞

$$\stackrel{\textcircled{L}}{=} \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{3x^2} \quad [\text{still } \frac{0}{0}]$$

$$\stackrel{\textcircled{L}}{=} \lim_{x \rightarrow 0} \frac{e^x - 1}{6x} \quad [\text{still } \frac{0}{0}]$$

$$\stackrel{\textcircled{L}}{=} \lim_{x \rightarrow 0} \frac{e^x}{6} = \frac{1}{6}$$

$$\boxed{\lim_{x \rightarrow 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3} = \frac{1}{6}}$$

3. Find the equation for the tangent line to the curve defined by the parametrisation $x = 1 + \frac{1}{t}$, $y = t^3 + 3$ (for $t > 0$), at the point $(x, y) = (2, 4)$.

(A) $y = -3x + 10$

(B) $y = 3x - 8$

(C) $y = -3x + 14$

(D) $y = 3x - 2$

(E) $y = -\frac{x}{3} + \frac{14}{3}$

(F) $y = -\frac{x}{3} + \frac{10}{3}$

$$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{-\frac{1}{t^2}} = -3t^4$$

$(x, y) = (2, 4)$ corresponds to $t = 1$.

$$m = -3(1)^4 = -3$$

The equation of the tangent line is:

$$y - 4 = -3(x - 2)$$

$$y - 4 = -3x + 6$$

$$\Rightarrow \boxed{y = -3x + 10}$$

4. Solve the initial value problem:

$$\frac{dx}{dt} + 2tx = x, x(0) = 5.$$

Use your solution to compute $x(3)$.

(A) $5e^{-6}$

(B) $5e^6$

(C) $6e^5$

(D) 3

(E) -10

(F) 0

$$\frac{dx}{dt} + (2t - 1)x = 0$$

The integrating factor is $e^{\int (2t-1) dt} = e^{t^2-t}$

$$e^{t^2-t} x' + (2t-1)e^{t^2-t} x = 0$$

$$= (e^{t^2-t} x)' = 0$$

$$\Rightarrow e^{t^2-t} x = C$$

Applying the initial condition: $e^{0^2-0} 5 = C \Rightarrow C = 5$

$$x(t) = \frac{5}{e^{t^2-t}} \quad \text{or} \quad \underline{x(t) = 5e^{t-t^2}}$$

$$x(3) = 5e^{-9+3} = 5e^{-6}$$

$$\boxed{x(3) = 5e^{-6}}$$

5. Evaluate the integral

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}} dx.$$

(A) $\pi/4$

(B) $\pi/2$

(C) π

(D) $2/3$

(E) $3/4$

(F) 1

let $1-x^2 = u$
 $x^2 = 1-u$; $x^3 = x \cdot x^2 = x \cdot (1-u)$
 $2x dx = -du \Rightarrow x dx = -\frac{du}{2}$

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}} dx = \frac{-1}{2} \int_1^0 \frac{(1-u) du}{\sqrt{u}} = -\frac{1}{2} \left[\int_1^0 u^{-1/2} du - \int_1^0 u^{1/2} du \right]$$

$$= -\frac{1}{2} \left[2u^{1/2} - \frac{2}{3} u^{3/2} \right]_1^0 = -\frac{1}{2} \left[- \left(2 - \frac{2}{3} \right) \right] = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}$$

$$\boxed{\int_0^1 \frac{x^3}{\sqrt{1-x^2}} dx = \frac{2}{3}}$$

6. Integrate

(A) -2

(B) $-3 \ln(2) + \ln(3)$

(C) $\ln(2)$

(D) $\pi/4$

(E) 0

(F) $\ln(3)$

$$\int_0^1 \frac{3x+2}{x^2-4} dx =$$

$$= \int_0^1 \frac{3x+2}{(x-2)(x+2)} dx =$$

$$= \int_0^1 \left(\frac{2}{x-2} + \frac{1}{x+2} \right) dx =$$

$$= \left[2 \ln|x-2| + \ln|x+2| \right]_0^1$$

$$= 2 \ln|-1| + \ln|3| - 2 \ln|-2| - \ln|2|$$

$$= \ln(3) - 3 \ln(2)$$

$$\boxed{\int_0^1 \frac{3x+2}{x^2-4} dx = -3 \ln(2) + \ln(3)}$$

$$\left\{ \begin{array}{l} x^2-4 = (x-2)(x+2) \\ \frac{3x+2}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \end{array} \right.$$

$$3x+2 = Ax+2A+Bx-2B$$

$$\left\{ \begin{array}{l} A+B=3 \quad (1) \\ 2(A-B)=2 \quad (2) \end{array} \right.$$

$$2(1) + (2) \rightarrow 4A = 8 \Rightarrow A = 2$$

$$(1): 2+B=3 \Rightarrow B=1$$

7. Find the limit of the sequence

$$a_n = \left\{ \frac{1}{2} \ln(n^2 + 1) - \ln(2n + 1) \right\}.$$

- (A) -2
- (B) $-\ln(2)$
- (C) $\ln(2)$
- (D) 0
- (E) 2
- (F) sequence diverges

First, let us put the a_n in a more convenient form:

$$a_n = \left\{ \ln(n^2 + 1)^{1/2} - \ln(2n + 1) \right\}$$

$$= \left\{ \ln \left(\frac{\sqrt{n^2 + 1}}{2n + 1} \right) \right\}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \ln \left(\frac{\sqrt{n^2 + 1}}{2n + 1} \right) = \ln \left(\lim_{n \rightarrow \infty} \sqrt{\frac{n^2 + 1}{4n^2 + 4n + 1}} \right)$$

$$= \ln \left(\sqrt{\lim_{n \rightarrow \infty} \frac{n^2 + 1}{4n^2 + 4n + 1}} \right) = \ln \left(\sqrt{\frac{1}{4}} \right) = \ln \left(\frac{1}{2} \right) = \cancel{\ln(1)} - \ln(2) = \boxed{-\ln 2}$$

8. Consider the sequence defined by

$$a_n = \left\{ \frac{(-1)^n + n}{(-1)^n - n} \right\}.$$

Does this sequence converge and, if it does, to what limit?

- (A) diverges
- (B) yes, to -1
- (C) yes, to 0
- (D) yes, to 1
- (E) yes, to 2
- (F) yes, to π

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{(-1)^n}{(-1)^n - n} \right) + \lim_{n \rightarrow \infty} \left(\frac{n}{(-1)^n - n} \right)$$

$$= 0 + (-1) = \boxed{-1}$$

9. Let

$$F(x) = \int_0^x e^{-t^2} dt.$$

Which of the following is the beginning of the Maclaurin series for F ?

(A) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

(B) $x - \frac{x^2}{2} + \frac{x^4}{6} - \frac{x^6}{24} + \dots$

(C) $x - \frac{x^3}{3} + \frac{x^5}{15} - \frac{x^7}{105} + \dots$

(D) $x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots$

(E) $x - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$

(F) $x + \frac{x^3}{3} + \frac{x^5}{15} + \frac{x^7}{105} + \dots$

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \quad \text{for all } t$$

$$e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots$$

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots$$

$$F(x) = \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots \right) dt = \left[t - \frac{t^3}{3} + \frac{t^5}{2! \cdot 5} - \frac{t^7}{3! \cdot 7} + \dots \right]_0^x$$

$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots$$

10. Determine if the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{3^n + 4^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{3}{7} \right)^n + \sum_{n=1}^{\infty} \left(\frac{4}{7} \right)^n$$

- (A) 1
- (B) 1/7
- (C) 5/12
- (D) 12/7
- (E) 25/12
- (F) divergent

Recall that $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

Both are convergent geometric series

$$= \frac{1}{1 - \frac{3}{7}} + \frac{1}{1 - \frac{4}{7}} - 2 \quad (-2 \text{ because our series starts at } n=1)$$

$$= \frac{1}{\frac{4}{7}} + \frac{1}{\frac{3}{7}} - 2$$

$$= \frac{7}{4} + \frac{7}{3} - 2 = \frac{21 + 28}{12} - 2 = \frac{49}{12} - \frac{24}{12} = \frac{25}{12}$$

$$\sum_{n=1}^{\infty} \frac{3^n + 4^n}{7^n} = \frac{25}{12}$$

11. Describe the curve defined by $x = \sin(t)$, $y = \sin^2(t)$?

- (A) a circle
- (B) an ellipse
- (C) an hyperbola
- (D) one branch of an hyperbola
- (E) a parabola
- (F) a part of a parabola

We see that $y = x^2$. This is a part of a parabola because both x and y are bounded while the parabola is unbounded.

12. Find the precise interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$$

(A) $(-1, 1]$

(B) $[-1, 1)$

(C) $(-\frac{1}{3}, \frac{5}{3}]$

(D) $[-\frac{1}{3}, \frac{5}{3})$

(E) $[0, \frac{4}{3}]$

(F) $(0, \frac{4}{3}]$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(3x-2)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(3x-2)^n} \right| =$$

$$= \frac{1}{3} |3x-2| < 1 \quad (\text{for convergence})$$

$$-3 < 3x-2 < 3$$

$$-1 < 3x < 5$$

$$-\frac{1}{3} < x < \frac{5}{3}$$

When $x = -\frac{1}{3}$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ convergent alternating series.

When $x = \frac{5}{3}$ we have $\sum_{n=1}^{\infty} \frac{3^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ divergent harmonic series.

The interval of convergence is $[-\frac{1}{3}, \frac{5}{3})$.

13. Which of the following tests will establish that the series

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{2n^5 + 1}}$$

Compare with

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5}} = \sum_{n=1}^{\infty} n^{-5/2}$$

$$= \sum_{n=1}^{\infty} n^{-3/2}$$

converges?

- (a) Comparison test with $\sum_{n=1}^{\infty} n^{-5/2}$
- (b) Comparison test with $\sum_{n=1}^{\infty} n^{-3/2}$
- (c) Comparison test with $\sum_{n=1}^{\infty} n^{-1/2}$

- (A) none
- (B) (a)
- (C) (b)**
- (D) (c)
- (E) (a),(b)
- (F) (a),(c)

14. Consider the following four series:

- (a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *Convergent p-series $p > 1$.*
- (b) $\sum_{n=1}^{\infty} 2^n$ *divergent*
- (c) $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ *divergent p-series $p < 1$*
- (d) $\sum_{n=1}^{\infty} (\frac{1}{2})^n$ *Convergent geometric series ($\frac{1}{2} < 1$).*

- (A) (a), (b), (c), and (d) all converge.
- (B) (a) and (b) converge, but (c) and (d) diverge.
- (C) (a) and (c) converge, but (b) and (d) diverge.
- (D) (a) and (d) converge, but (b) and (c) diverge.**
- (E) (a) converges, but (b), (c), and (d) diverge.
- (F) (a) diverges, but (b), (c), and (d) converge.

15. Find the area *one leaf* (i.e. one loop) of the graph of $r = 4 \cos(3\theta)$.

(A) $2\pi/3$

(B) $3\pi/4$

(C) $\pi/2$

(D) $4\pi/3$

(E) $7\pi/4$

(F) 2π

Will compute area of leaf I.

$$r=0 \quad \text{when} \quad \cos(3\theta)=0 \Rightarrow 3\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{6}$$

I will compute the area of the shaded region and use symmetry to give the area of one leaf.

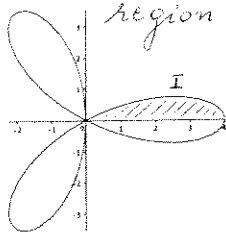


Figure 1: $r = 4 \cos(3\theta)$

$$\begin{aligned} A_{\text{one leaf}} &= 2 \left[\frac{1}{2} \int_0^{\pi/6} (4 \cos 3\theta)^2 d\theta \right] = \int_0^{\pi/6} 16 \cos^2 3\theta d\theta = \frac{16}{2} \int_0^{\pi/6} (1 + \cos 6\theta) d\theta \\ &= 8 \left(\theta + \frac{1}{6} \sin 6\theta \right) \Big|_0^{\pi/6} = 8 \cdot \frac{\pi}{6} = \boxed{\frac{4\pi}{3}} \end{aligned}$$

16. (Bonus) What is the largest open interval on which the series

$$\sum_{n=1}^{\infty} \frac{n(x-5)^n}{n^3 + 4^n}$$

converges?

(A) $(-4, 4)$

(B) $(-4, 5)$

(C) $(4, 6)$

(D) $(1, 9)$

(E) $(4, 5)$

(F) $(-4, 14)$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-5)^{n+1}}{(n+1)^3 + 4^{n+1}} \cdot \frac{n^3 + 4^n}{n(x-5)^n} \right|$$

$$= |x-5| \lim_{n \rightarrow \infty} \left(\frac{(n+1)}{n} \cdot \frac{n^3 + 4^n}{(n+1)^3 + 4^{n+1}} \right) = \frac{1}{4}$$

$$= \frac{|x-5|}{4} < 1$$

for convergence

$$-4 < x-5 < 4$$

$\boxed{1 < x < 9}$ is the largest open interval on which the above series converges.