University of Utah
Department of Mathematics
Exam 2

April 3, 2009
Calculus II (MATH 1220)
Name: SOLUTIONS

Question	Score	Maximum
1		3
2		3
3		3
4		3
5		3
6		3
7		3
8		3
9		3
10		3
11		3
TOTAL		30

Instructions

1. The exam lasts 50 minutes. Do not open this booklet, nor start before you are given permission to.
2. There are 10 questions worth 3 points + an $11th$ question worth 3 points as well. The $11th$ question will count as a bonus question.
3. Write your answers *unambiguously*!
4. <u>Show all relevant work</u>, no credit will be given for a guessed correct answer.
5. No calculators, notes or books can be used.
6. You can use the last page as scratch paper.

Compute the following limits:

1. $\lim_{x\to\infty} x^{\frac{1}{x}}$ $[\infty^0 \text{ form}]$

$$x^{\frac{1}{x}} = e^{\ln x^{\frac{1}{x}}} = e^{\frac{1}{x}\ln x}$$

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{L}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\lim_{x\to\infty} e^{0} = 1$$

$$\boxed{\text{Thus, } \lim_{x\to\infty} x^{\frac{1}{x}} = 1}$$

2. $\lim_{x\to\infty} \frac{(\ln x)^2}{2^x} \overset{L}{=} \lim_{x\to\infty} \frac{2\ln x\cdot\frac{1}{x}}{2^x \ln 2} = \lim_{x\to\infty} \frac{2\ln x}{2^x\cdot x\cdot \ln 2}$

$$\overset{L}{=} \lim_{x\to\infty} \frac{2\cdot\frac{1}{x}}{x\cdot 2^x(\ln 2)^2 + 2^x(\ln 2)} = \lim_{x\to\infty} \frac{2}{x\cdot 2^x(\ln 2)\ (x\ln 2+1)} = 0$$

$$\boxed{\lim_{x\to\infty} \frac{(\ln x)^2}{2^x} = 0}$$

Evaluate the following integrals:

3. $\int_{-\infty}^{1} e^{2x}dx = \lim_{a\to-\infty} \int_a^1 e^{2x}dx = \lim_{a\to-\infty} \left(\frac{1}{2}e^{2x}\right)\Big|_a^1$

$$= \lim_{a\to-\infty} \frac{1}{2}\left(e^2 - \underbrace{e^{2a}}_{\substack{\to 0 \\ \text{as } a\to-\infty}}\right) = \boxed{\frac{e^2}{2}}$$

4. $\int_0^1 \frac{x}{\sqrt{1-x^2}}dx = \lim_{b\to 1}\int_0^b \frac{x}{\sqrt{1-x^2}}dx \overset{(u\text{-substitution})}{=} -2\lim_{b\to1}\int_1^{1-b^2}\frac{1}{\sqrt{u}}du$

$$= -\frac{1}{\cancel{2}}\lim_{b\to1}\left(\cancel{2}\,u^{1/2}\right)\Big|_1^{1-b^2} = -\lim_{b\to1}\left(\underbrace{(1-b^2)^{1/2}}_{\substack{\to 0 \\ \text{as } b\to1}} - 1\right) = \boxed{1}$$

5. Determine whether the following sequence converges or diverges.
$a_n = e^{-n} \sin n + 1$

$$-1 \leq \sin n \leq 1$$

$$-\frac{1}{e^n} \leq \frac{\sin n}{e^n} \leq \frac{1}{e^n}$$

$$\lim_{n\to\infty} -\frac{1}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0$$

By squeeze theorem, $\lim_{n\to\infty} e^{-n} \sin n = 0$

Thus, $\lim_{n\to\infty} e^{-n} \sin n + 1 = 1.$

The sequence converges.

6. In the following problems [6 – 8], do the following series converge or diverge? Find the convergence set as well as the radius of convergence where applicable.
$$\sum_{n=1}^{\infty} \frac{3^n n!}{(n+2)!}$$

Let's use the Ratio Test.

$$\lim_{n\to\infty} \left[\frac{a_{n+1}}{a_n}\right] = \lim_{n\to\infty} \left[\frac{3^{\cancel{n+1}}(n+1)\cancel{!}}{(n+3)\cancel{!}} \cdot \frac{\cancel{(n+2)!}}{\cancel{3^n}\,\cancel{n!}}\right]$$

$$= \lim_{n\to\infty} \left[\frac{3(n+1)}{(n+3)}\right] = 3 > 1$$

The series diverges.

7. $\sum_{n=1}^{\infty} \frac{n}{1+n}$

$$\lim_{n\to\infty} \frac{n}{1+n} = 1$$

The series diverges.

8. $\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n+1}$

Again, let us use Absolute Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{2^{n+1}+1} \cdot \frac{2^n+1}{(x-3)^n} \right|$$

$$= \lim_{n\to\infty} |x-3| \cdot \frac{1}{2} = \frac{1}{2}|x-3|$$

For the series to converge, $\frac{1}{2}|x-3| < 1$

$$-2 < x-3 < 2$$

$$1 < x < 5$$

(a) when $x=1$, $\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n+1}$ is $\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n+1}$ and this series diverges (absolute ratio test)

(b) when $x=5$, $\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n+1}$ is $\sum_{n=0}^{\infty} \frac{2^n}{2^n+1}$ this series diverges (n^{th} limit test)

The convergence set for this series is $(1,5)$ and the radius of convergence is 2.

9. Find a power series representation for $\frac{1}{(1-x)^2}$.

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = (1 + x + x^2 + x^3 + x^4 + \cdots)'$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty} (n+1) x^n}$$

10. Find the Taylor polynomial of order 3 based at $a = 2$ for $\sqrt{x}$.

$f^{(0)}(x) = (x)^{1/2}$ at $a = 2$ $\quad f^{0}(2) = \sqrt{2}$

$f'(x) = \frac{1}{2}(x)^{-1/2}$ -"- -"- $\quad f'(2) = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

$f''(x) = -\frac{1}{4} x^{-3/2}$ -"- -"- $\quad f''(2) = -\frac{1}{4}\frac{1}{2\sqrt{2}} = -\frac{1}{8\sqrt{2}} = -\frac{\sqrt{2}}{16}$

$f'''(x) = \frac{3}{8} x^{-5/2}$ -"- -"- $\quad f'''(2) = \frac{3}{8}\frac{1}{4\sqrt{2}} = \frac{3}{32\sqrt{2}} = \frac{3\sqrt{2}}{64}$

$$\boxed{T_3(x) = \sqrt{2} + \frac{\sqrt{2}}{4}(x-2) - \frac{\sqrt{2}}{32}(x-2)^2 + \frac{\sqrt{2}}{128}(x-2)^3}$$

11. Determine whether the following series converges or diverges. If it converges, find the limit.

$$\sum_{k=0}^{\infty}\left(\frac{3}{2^k}+\frac{4}{3^k}\right) = \underbrace{3\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k + 4\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k}_{\text{geometric series}}$$

(Both $\left|\frac{1}{2}\right|$ and $\left|\frac{1}{3}\right|$ are less than 1, thus they both converge.)

$$= 3\cdot\frac{1}{1-\frac{1}{2}} + 4\cdot\frac{1}{1-\frac{1}{3}}$$

$$= 6+6$$

$$= \boxed{12}$$

Scratch Paper