University of Utah
Department of Mathematics
Exam 1

February 20, 2009
Calculus II (MATH 1220)
Name: SOLUTIONS

Question	Score	Maximum
1		3
2		3
3		3
4		3
5		3
6		3
7		3
8		3
9		3
10		3
Bonus		4
TOTAL		30

Instructions

1. The exam lasts 50 minutes. Do not open this booklet, nor start before you are given permission to.
2. There are 10 questions worth 3 points + a bonus question worth 4 points.
3. Write your answers *unambiguously*!
4. Show all relevant work, no credit will be given for a guessed correct answer.
5. No calculators, notes or books can be used.
6. You can use the last page as scratch paper.

Compute the derivatives of the following functions:

1. $y(t) = \frac{t^2+18t}{\sqrt[5]{t^5-4}}$

Take the ln of both sides:

$$\ln(y(t)) = \ln\left(\frac{t^2+18t}{\sqrt[5]{t^5-4}}\right)$$

$$\ln(y(t)) = \ln(t^2+18t) - \frac{1}{5}\ln(t^5-4)$$

Now differentiate both sides:

$$\frac{y'(t)}{y(t)} = \frac{2t+18}{t^2+18t} - \frac{1}{\cancel{5}}\,\frac{\cancel{5}t^4}{t^5-4}$$

$$\Rightarrow y'(t) = \left(\frac{2t+18}{t^2+18t} - \frac{t^4}{t^5-4}\right)\frac{t^2+18t}{\sqrt[5]{t^5-4}}$$

2. $y(x) = e^{x^3 \ln(x)}$

$$y'(x) = (x^3 \ln x)'\, e^{x^3 \ln x}$$

$$= \left(\frac{x^3}{x} + 3x^2 \ln x\right) e^{x^3 \ln x}$$

$$= x^2 (1 + 3\ln x)\, x^{x^3}$$

3. $y(x) = 2^{\cos(x)} + x^{\pi^2+1}$

$$y'(x) = \left(e^{\ln 2^{\cos x}}\right)' + \left(x^{\pi^2+1}\right)'$$

$$= \left(e^{\cos x \ln 2}\right)' + \left(x^{\pi^2+1}\right)'$$

$$= (\cos x \ln 2)' \, e^{\cos x \ln 2} + (\pi^2+1)\, x^{\pi^2}$$

$$= -\ln 2 \sin x \, 2^{\cos x} + (\pi^2+1)\, x^{\pi^2}$$

//

4. $y = x^{\sin(x)+2x}$

$$y'(x) = \left(e^{(\sin x + 2x)\ln x}\right)'$$

$$= \left[(\sin x + 2x)\ln x\right]' e^{(\sin x + 2x)\ln x}$$

$$= \left(\frac{\sin x + 2x}{x} + \ln x \cdot (\cos x + 2)\right) x^{\sin x + 2x}$$

//

5. Solve the following differential equation:

$$\frac{dy}{dx} = x(e - 2y)$$

$y = \frac{e}{2}$ when $x = 0$.

$y' + 2xy = xe$

(a) The integrating factor is: $e^{\int 2x\,dx} = e^{x^2}$

(b) Multiply both sides by the integrating factor.

$$\underbrace{e^{x^2} y' + 2x e^{x^2} y}_{\parallel} = x e^{x^2+1}$$

$$(e^{x^2} y)' = x e^{x^2+1}$$

(c) Integrate both sides:

$$e^{x^2} y = \int x e^{x^2+1} dx \qquad \text{let } x^2+1 = u$$

$$2x\,dx = du$$

$$e^{x^2} y = \int \frac{e^u du}{2} \qquad x\,dx = \frac{du}{2}$$

$$e^{x^2} y = \frac{1}{2} e^{x^2+1} + C$$

(d) Apply initial conditions:

$$\frac{e}{2} = \frac{1}{2} e + C \Rightarrow \boxed{C = 0}$$

(e) The solution to the differential equation is: $y = \frac{e}{2}$ //

6. Compute $(f^{-1})'(1)$ for $f(x) = x^3 - 3x^2 + 1$

By the inverse function theorem:

$$(f^{-1})'(1) = \frac{1}{f'(0)} \qquad (x = 0, 3 \text{ for } y = 1)$$

but $f'(0) = 0$ (cannot invert f at 0).

$$= \frac{1}{3x^2 - 6x\Big|_{x=3}} = \frac{1}{27-18} = \frac{1}{9} \quad //$$

Compute the following integrals:

7. $\int x \cos(\frac{x}{2}) dx$ Integrate by parts: let $x = u$, $\cos\left(\frac{x}{2}\right) dx = d$

$dx = du$, $2 \sin\left(\frac{x}{2}\right) = v$

$= 2x \sin\left(\frac{x}{2}\right) - 2\int \sin\left(\frac{x}{2}\right) dx$

$= 2x \sin\left(\frac{x}{2}\right) + 4 \cos\left(\frac{x}{2}\right) + C$ //

8. $\int \frac{2x-4}{(x-2)^2(2-x)} dx = \int - \frac{2(\cancel{x-2})}{(x-2)^2 (\cancel{x-2})} dx = -2\int \frac{1}{(x-2)^2} dx$

$= \frac{2}{x-2} + C$ //

9. $\int \sin^3(x)dx = \int \sin x \cdot \sin^2 x \, dx = \int \sin x \, (1 - \cos^2 x) \, dx$

$\left\{ \begin{array}{l} \text{let } \cos x = u \\ \quad -\sin x \, dx = du \end{array} \right.$

$= -\int (1 - u^2) \, du$

$= \frac{u^3}{3} - u + C$

$= \frac{\cos^3 x}{3} - \cos x + C$ //

10. $\int e^{\cos(x)} \sin(x)dx = -\int e^u \, du$

$\left\{ \begin{array}{l} \text{let } \cos x = u \\ \quad -\sin x \, dx = du \end{array} \right.$

$= -e^u + C$

$= -e^{\cos x} + C$ //

11. (Bonus) Find $\int \sin(x)e^x dx$.
Hint: Integrate twice by parts.

1) let $\sin x = u$ $\qquad e^x dx = dv$

$\cos x \, dx = du$ $\qquad e^x = v$

$$\int \sin x \, e^x dx = e^x \sin x - \int e^x \cos x \, dx$$

2) let $\cos x = u$ $\qquad e^x dx = dv$

$-\sin x \, dx = du$ $\qquad e^x = v$

$$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right]$$

So far we have:

$$\int \sin x \, e^x dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$\iff$

$$2 \int \sin x \, e^x dx = e^x \sin x - e^x \cos x$$

$$\Rightarrow \int \sin x \, e^x dx = \frac{e^x (\sin x - \cos x)}{2} + C \; //$$

Scratch Paper