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Linear Algebra (MATH 2270-001)
October 2, 2009
Exam 1

Name: Solutions

Question	Score	Max	Question	Score	Max
1		5	4		2
2		5	☐ 5		6
3		8	☐ 6		6
TOTAL					26

Instructions

1. The exam lasts 50 minutes. Do not open this booklet, nor start before you are given permission to.
2. This exam consists of 6 questions. You should answer questions 1 – 4 and choose between questions 5 and 6. I will *only* grade one of them, so please check the box of the problem that you wish me to grade.
3. Write your answers unambiguously by *showing all relevant work*.
4. No calculators, notes or books can be used.
5. You can use the last sheet of this booklet for lengthy computations. Note that I will not refer to this page while grading.
6. Please make sure that any electronic device that you possess is turned off before you enter the exam room.
7. Relax and good luck!

1. (a) (3 pts) Find the inverse of

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 1 & -1 & 0 \\ 4 & 1 & 3 \end{pmatrix}$$

$$\left(\overbrace{\begin{matrix} 3 & 0 & 2 \\ 1 & -1 & 0 \\ 4 & 1 & 3 \end{matrix}}^{A} \middle| \overbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}^{I}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{matrix} 1 & -1 & 0 \\ 4 & 1 & 3 \\ 3 & 0 & 2 \end{matrix} \middle| \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}\right) \xrightarrow[3R_1 - R_3]{4R_1 - R_2} \left(\begin{matrix} 1 & -1 & 0 \\ 0 & -5 & -3 \\ 0 & -3 & -2 \end{matrix} \middle| \begin{matrix} 0 & 1 & 0 \\ 0 & 4 & -1 \\ -1 & 3 & 0 \end{matrix}\right)$$

$$\xrightarrow[-3R_2 + 5R_3]{-\frac{1}{5}R_2 + R_1} \left(\begin{matrix} 1 & 0 & \frac{3}{5} \\ 0 & -5 & -3 \\ 0 & 0 & -1 \end{matrix} \middle| \begin{matrix} 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 4 & -1 \\ -5 & 3 & 3 \end{matrix}\right) \xrightarrow[\frac{3}{5}R_3 + R_1]{-3R_3 + R_2} \left(\begin{matrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{matrix} \middle| \begin{matrix} -3 & 2 & 2 \\ 15 & -5 & -10 \\ -5 & 3 & 3 \end{matrix}\right) \xrightarrow[(-1)R_3]{(-\frac{1}{5})R_2} \left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \middle| \underbrace{\begin{matrix} -3 & 2 & 2 \\ -3 & 1 & 2 \\ 5 & -3 & -3 \end{matrix}}_{A^{-1}}\right)$$

<u>Check</u>:

$$\begin{pmatrix} 3 & 0 & 2 \\ 1 & -1 & 0 \\ 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} -3 & 2 & 2 \\ -3 & 1 & 2 \\ 5 & -3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\text{OR}}{=} \begin{pmatrix} -3 & 2 & 2 \\ -3 & 1 & 2 \\ 5 & -3 & -3 \end{pmatrix}\begin{pmatrix} 3 & 0 & 2 \\ 1 & -1 & 0 \\ 4 & 1 & 3 \end{pmatrix}$$

(b) (2 pts) Use (a) to solve the following system:

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 & 2 & 2 \\ -3 & 1 & 2 \\ 5 & -3 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ -10 \end{pmatrix}$$

so $\begin{cases} x = 7 \\ y = 5 \\ z = -10 \end{cases}$ $\quad$ sol $= \{(7, 5, -10)\}$

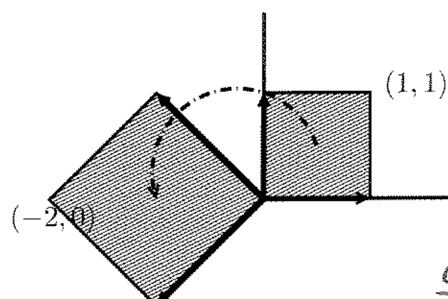
2. (a) (2 pts) Find a 3×3 matrix such that $A^3 = I$, $A^2 \neq I$.
Hint: Think of geometric transformations.

One possibility:

$$\begin{pmatrix} \cos 120 & -\sin 120 & 0 \\ \sin 120 & \cos 120 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Later, we'll show that any such matrix is a rotation by 120° around some axis in $\mathbb{R}^3$.

(b) (3 pts) Find a matrix associated to the linear transformation sending the unit square to the depicted square:
Note: There are 2 possibilities.



Notice that the transformation is the composition of a rotation by 135° and a scaling by a factor of $\sqrt{2}$.

The matrix is thus:

$$\sqrt{2}\begin{pmatrix} \cos 135 & -\sin 135 \\ \sin 135 & \cos 135 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ +1 & -1 \end{pmatrix}$$

OR

Notice that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

Placing those in columns we have: $\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$

I did here the case in which the orientation is preserved. The other case, can be treated similarly.

3. (8 pts) Consider the map: $T : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (y, 2z, 0)$

(i) (3 pts) Show that T is linear and find the associated matrix A_T.

(ii) (2 pts) Find a basis for the kernel of T. What is its dimension?

(iii) (2 pts) Find a basis for the image of T. What is its dimension?

(iv) (1 pt) Check the Rank-Nullity theorem in this case.

(i) Pick $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$.

- (a) $T(\vec{u}) = (u_2, 2u_3, 0)$

$T(\vec{v}) = (v_2, 2v_3, 0)$

$$T(\vec{u}) + T(\vec{v}) = (u_2, 2u_3, 0) + (v_2, 2v_3, 0)$$
$$= (u_2 + v_2,\ 2(u_3 + v_3), 0)$$
$$= T(\vec{u} + \vec{v}) \checkmark$$

(b) $$kT(\vec{u}) = k(u_2, 2u_3, 0)$$
$$= (ku_2, 2ku_3, k0)$$
$$= T(k\vec{u}) \checkmark$$

$\therefore$ T is linear.

- $$A_T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

(ii) $$\ker T = \{(x, y, z) \mid (y, 2z, 0) = (0, 0, 0)\}$$
$$= \{(x, y, z) \mid y = 0, z = 0\}$$
$$= \{(x, 0, 0)\} = \{x(1, 0, 0)\}, \quad \text{basis} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$\cdot \dim(\ker A_T) = 1$

(iii) $$\operatorname{im} T = \operatorname{Span}\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \quad \text{basis} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\rangle$$

$\cdot \dim(\operatorname{im} A_T) = 2 = \operatorname{rk}(A_T)$

(iv) $$\dim(\operatorname{im} A_T) + \dim(\ker A_T) = \dim \mathbb{R}^3$$
$$1 + 2 = 3 \checkmark$$

4. (2 pts) A matrix A is called idempotent if $A^2 = A$. Show that if A is idempotent, then so is $I - A$.

$$
\begin{aligned}
(I-A)^2 &= (I-A)(I-A) \\
&= I^2 - IA - AI + A^2 \\
&= I^2 - A - A + A^2 \\
&= I^2 - A - \cancel{A} + \cancel{A} \quad \text{(since } A \text{ is idempotent)} \\
&= I - A
\end{aligned}
$$

Hence $I - A$ is idempotent. $\longrightarrow \square$

5. (6 pts) Find all matrices that commute with $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. Discuss in terms of k.

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a+kc & b+kd \\ c & d \end{pmatrix} = \begin{pmatrix} a & ak+b \\ c & ck+d \end{pmatrix}$$

$$\begin{cases} a+kc = a \\ b+kd = ak+b \\ c = c \\ d = ck+d \Rightarrow kc=0 \end{cases} \iff \begin{cases} kc = 0 \\ (a-d)k = 0 \end{cases} \quad (*)$$

We distinguish 2 cases:

$\longrightarrow$ $\underline{k=0}$

$(*)$: $\left\{ 0 = 0 \right.$

The system puts no constraint on a, b, c, d. Hence $sol = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\}$.

<u>Remark</u>, indeed we know that the identity commutes with all matrices.

$\longrightarrow$ $\underline{k \neq 0}$

$$(*) = \begin{cases} c = 0 \\ a = d \end{cases}$$

Hence $sol = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right\}$.

6. (6 pts) Let A and B be two $n \times n$ matrices. Are the following statements true or false. *Justify your answers.*

(i) (3 pts) $\text{im}(AB) \subset \text{im}(A)$

(ii) (3 pts) $\ker(B) \subset \ker(AB)$

(i) True. Take $\vec{y} \in im(AB)$.
There exists $\vec{x}$ s.t. $\vec{y} = (AB)\vec{x}$
or, by associativity, $\vec{y} = A(B\vec{x})$
i.e. $\vec{y}$ is in the image of A.

(ii) True. Take $\vec{x} \in ker(B)$.
Now, $(AB)\vec{x} = A(B\vec{x}) = A \cdot \vec{0} = \vec{0}$
Hence, $\vec{x}$ is in the kernel of AB.

Scrap Paper: