Section B.1: Doubling Time

Definition of doubling time

The time required for each doubling in exponential growth is called the *doubling time*. After a time t, an exponentially growing quantity with a doubling time T_{double} increases in size by a factor of

$$2^{\frac{t}{T_{double}}}$$

The new value of the growing quantity is related to its initial value (at $t = 0$) by

$$\text{new value} = \text{initial value} \times 2^{\frac{t}{T_{double}}}$$

Ex.1

Consider an initial population of $10,000$ that grows with a doubling time of 10 years. What is the population after 10, 20, 30, 50 years?

- In 10 years the population increases by a factor of 2:
- In 20 years the population increases by a factor of $4 = 2^2$:
- In 30 years the population increases by a factor of $8 = 2^3$:
- In 50 years:

Ex.2
Compound interest is a form of exponential growth because an interest bearing account grows by the same percentage each year. Suppose your bank account has a doubling time of 13 years, by what factor does your balance increase in 50 years?

Ex.3 World population growth.
World population doubled from 3 billion in 1960 to 6 billion in 2000. Suppose that world population continues to grow with a doubling time of 40 years. What will the population be in 2030? in 2200?

Section B.2: The Approximate Doubling Time Formula

Definition of approximate doubling time formula
For a quantity growing exponentially at a rate of $P\%$ per time period, the doubling time is approximately

$$T_{double} \sim \frac{70}{P}$$

This approximation works best for small growth rates and breaks down for growth rates over about 15% and it is called rule of 70.

Ex.4
World population was about 6.0 billion in 2000 and was growing at a rate of about 1.4% per year. What is the approximate doubling time at this growth rate? If this growth rate continues, what will the population be in 2030?

Ex.5
World population doubled in the 40 years from 1960 to 2000. What was the average percentage growth rate during this period? Contrast this growth rate with the 2000 growth rate of 1.4% per year.

Section B.3: Exponential Decay and Half-Life

Exponential decay and half-life

Exponential decay occurs whenever a quantity decreases by the same percentage in every fixed time period. In that case, the value of the quantity repeatedly decreases to half its value, with each halving occurring in a time called the *half-life* $= T_{half-life}$.

After a time t, an exponentially decaying quantity with a half-time time $T_{half-life}$ decreases in size by a factor of

$$\left(\frac{1}{2}\right)^{\frac{t}{T_{half-life}}}$$

The new value of the decaying quantity is related to its initial value (at $t = 0$) by

$$\text{new value} \;=\; \text{initial value} \;\times\; \frac{1}{2}^{\frac{t}{T_{half-life}}}$$

Ex.6

You may have heard half-lives described for radioactive materials such as uranium or plutonium. For example, radioactive plutonium-239 (Pu-239) has a half-life of about $24,000$ years. Suppose that 100 pounds of Pu-239 is deposited at a nuclear waste site. How much plutonium we have after $24,000$ years? And after $48,000$ years? And after $72,000$ years?

Ex.7 Carbon-14 decay.
Radioactive carbon-14 has a half-life of about 5700 years. It collects in organisms only while they are alive. Once they are dead, it only decays. What fraction of the carbon-14 in an animal bone still remains 1000 years after the animal has died?

Ex.8 Plutonium after $100,000$ years.
Suppose that 100 pounds of Pu-239 is deposited at a nuclear waste site. How much of it will still be present in $100,000$ years?

Section B.4: The Approximate Half-Life Formula

Definition of approximate half-life formula

The approximate doubling time formula (the rule of 70) found earlier works equally well for exponential decay if we replace the doubling time with the half-life and the percentage growth rate with the percentage decay rate. For a quantity decaying exponentially at a rate of $P\%$ per time period, the half-life is approximately

$$T_{half-life} \sim \frac{70}{P}$$

This approximation works best for small decay rates and breaks down for decay rates over about 15%.

Ex.9

Suppose that inflation causes the value of the Russian ruble to fall at a rate of 12% per year (relative to the dollar). Approximately how long does it take for the ruble to lose half its value?

Section B.5: Exact Formulas for Doubling Time and Half-Life

Exact formulas for doubling time and half-life

The approximate doubling time and half-life formulas are useful because they are easy to remember. However, for more precise work or for cases of larger rates where the approximate formulas break down, there are exact formulas.

We define the *fraction growth rate* as $r = \frac{P}{100}$, with r positive for growth and negative for decay. For example, if the percentage growth rate is 5% per year, the fractional growth rate is $r = 0.05$ per year. For a 5% decay rate per year, the fractional growth rate is $r = -0.05$ per year.

For an exponentially growing quantity with a fractional growth rate r, the doubling time is

$$T_{double} = \frac{\log_{10}(2)}{\log_{10}(1 + r)}$$

For an exponentially decaying quantity with a fractional growth rate r, the half-life is

$$T_{half-life} = -\frac{\log_{10}(2)}{\log_{10}(1 + r)}$$

Note that the units of time used for T and r must be the same.

Ex.10 Large growth rate.

A population of rats is growing at a rate of 80% per month. Find the exact doubling time for this growth rate and compare it to the doubling time with the approximate doubling time formula.

Ex.11
Suppose the Russian ruble is falling in value against the dollar at 12% per year. Using the exact half-life formula, determine how long it takes the ruble to lose half its value.