Section B.1: Simple versus Compound Interest

Simple interest and compound interest

The *principal* in financial formulas is the balance upon which interest is paid. *Simple interest* is interest paid only on the original principal, and not on any interest added at later dates.
Compound interest is interest paid both on the original principal and on all interest that has been added to the original principal.

Ex.1
You deposit $1000 in Honest John's Money Holding Service, which promises to pay 5% interest each year. At the end of the first year, Honest John's sends a check for

After 3 years you receive total interest of

Your original $1000 has grown in value to $1150. This is an example of simple interest.

Ex.2
Suppose that you place the $1000 in a bank account that pays the same 5% interest each year as in Example 1, but instead of paying you the interest directly, the bank adds the interest to your account. At the end of the first year, you have

After the second year you have

After the third year you have

This is an example of compound interest. Despite identical interest rates, you end up with $7.63 more if you use the bank instead of Honest John's.

Ex.3 Savings bond.
While banks almost always pay compound interest, bonds usually pay simple interest. Suppose you invest $\$1000$ in a savings bond that pays simple interest of 10% per year. How much total interest will you receive in 5 years? If the bond paid compound interest would you receive more or less total interest? Explain.

Section B.2: Compound Interest Formula

Compound interest formula

When interest is compounded just once a year, the interest rate is called the *annual percentage rate* (APR)

$$A = P \times (1 + \text{ APR})^Y$$

where

$A =$ accumulated balance after Y years

$P =$ starting principal

$APR =$ annual percentage rate (as a decimal)

$Y =$ number of years

Be sure to note that the annual interest rate (APR) should always be expressed as a decimal rather than a percentage.

Ex.4

From Example 3 we get:

AFTER N YEARS	INTEREST	BALANCE
1	$10\% \times 1000 = 100$	$1000 + 100 = 1100 = 1000 \times (1.1)$
2	$10\% \times 1100 = 110$	$1100 + 110 = 1210 = 1000 \times (1.1)^2$
3	$10\% \times 1210 = 121$	$1210 + 121 = 1331 = 1000 \times (1.1)^3$
4	$10\% \times 1331 = 133.1$	$1331 + 133.1 = 1464.1 = 1000 \times (1.1)^4$
5	$10\% \times 1464.1 = 146.41$	$1464.1 + 146.41 = 1610.51 = 1000 \times (1.1)^5$

So in this case:

Ex.5 Simple and compound interest.
You invest $\$100$ in two accounts that each pay an interest rate of 10% per year. However, one account pays simple interest and one account pays compound interest. Make a table that shows the growth of each account over a 5 year period. Use the compound interest formula to verify the result in the table for the compound interest case.

	SIMPLE INTEREST ACCOUNT		COMPOUND INTEREST ACCOUNT	
End of year	Interest Paid	New Balance	Interest Paid	New Balance
1	$10\% \times 100 = 10$	$100 + 10 = 110$	$10\% \times 100 = 10$	$100 + 10 = 110$
2	$10\% \times 100 = 10$	$110 + 10 = 120$	$10\% \times 110 = 11$	$110 + 11 = 121$
3	$10\% \times 100 = 10$	$120 + 10 = 130$	$10\% \times 121 = 12.1$	$121 + 12.1 = 133.1$
4	$10\% \times 100 = 10$	$130 + 10 = 140$	$10\% \times 133.1 = 13.31$	$133.1 + 13.1 = 146.41$
5	$10\% \times 100 = 10$	$140 + 10 = 150$	$10\% \times 1464.1 = 14.64$	$146.41 + 14.64 = 161.05$

Section B.3: Compound Interest as Exponential Growth

Ex.6
If we consider $Y = 100$, $P = \$100$, APR= 10%, then the accumulated balance is

If we connect all the values of A in a smooth way starting at time 0 till 100 years we get

Note that the value rises much more rapidly in later years. This rapid growth is a hallmark of what we generally call *exponential growth*.

Ex.7 Mattress investments.
Your grandfather put $\$100$ under his mattress 50 years ago. If he had instead invested it in a bank account paying 3.5% interest compounded yearly, how much would it be worth now?

Section B.4: Compound Interest Paid More Than Once a Year

Compound interest paid more than once a year

If an interest is paid n times per year, the interest rate at each payment is APR. The total number of times that interest is paid after Y years is nY. We therefore find the following formula for interest paid more than once each year:

$$A = P \times \left(1 + \frac{\text{APR}}{n}\right)^{(nY)}$$

where
$A =$ accumulated balance after Y years
$P =$ starting principal
$APR =$ annual percentage rate (as a decimal)
$n =$ number of compounding periods per year
$Y =$ number of years
Note that Y is not necessarily an integer. For example, a calculation for three and half years would have $Y = 3.5$.

Ex.8
Suppose you deposit \$1000 into a bank that pays compound interest at an annual percentage rate of APR $= 8\%$. If your interest is paid all at once at the end of the year, you will receive interest of

Your year balance will be

Now, assume that the bank pays the interest quarterly, or four times a year (that is, once every 3 months). The quarterly interest rate is one-fourth of the annual interest rate:

The table shows how quarterly compounding affects the \$1000 starting principal during the first year.

AFTER N QUARTERS	INTEREST PAID	NEW BALANCE
1st quarter (3 months)	$2\% \times 1000 = 20$	$1000 + 20 = 1020$
2nd quarter (6 months)	$2\% \times 1020 = 20.40$	$1020 + 20.40 = 1040.40$
3rd quarter (9 months)	$2\% \times 1040 = 20.81$	$1040.40 + 20.81 = 1061.21$
4st quarter (1 year)	$2\% \times 1061.21 = 21.22$	$1061.21 + 21.22 = 1082.43$

Ex.9 Monthly compounding at 3%.
You deposit $\$5000$ in a bank account that pays an APR of 3% and compound interest monthly. How much money will you have after 5 years? Compare this amount to the amount you would have if interest were paid only once per year.

Section B.5: Annual Percentage Yield (APY)

Annual percentage yield

The *annual percentage yield* (APY) is the actual percentage by which a balance increases in one year:

$$\text{APY} = \text{ relative increase } = \frac{\text{absolute increase}}{\text{starting principal}}$$

It is equal to the APR if interest is compounded annually. It is greater than the APR if interest is compounded more than once a year.

The APY does not depend on the starting principal.

Ex.10 More compounding means a higher yield.

You deposit $\$1000$ into an account with APR $= 8\%$. Find the annual percentage yield with monthly compounding and with daily compounding.

Section B.6: Continuous Compounding

Continuous compounding

If we could compound infinitely many times per year (more often than every second or every trillionth of a second), the annual yield would not go over a limit.

Compounding infinitely many times per year is called *continuous compounding*. It represents the best possible compounding for a particular APR. With continuous compounding, the compound interest formula takes the following special form:

$$A = P \times e^{(\mathrm{APR} \times Y)}$$

where

$A =$ accumulated balance after Y years

$P =$ starting principal

$APR =$ annual percentage rate (as a decimal)

$Y =$ number of years

The number e is a special irrational number with a value of $e \sim 2.71828$.

Ex.11

You deposit $\$100$ in an account with an APR of 8% and continuous compounding. How much will you have after 10 years?

Section B.7: Planning Ahead with Compound Interest

Ex.12
Suppose you could make a single deposit in an investment with an interest rate of APR $= 5\%$, compounded annually, and leave it there for the next 18 years. How much would you have to deposit now to realize $\$100,000$ after 18 years?

Ex.13
Repeat Example 12, but with an interest rate of APR $= 7\%$ and monthly compounding. Compare the results.