Making Minimal Surfaces with Complex Analysis

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March 26, 2002

Abstract:

In 1866 Karl Weierstrass discovered an amazing connection between the shapes of soap films and the field of complex analysis. Starting with the microscopic characterizations of minimal surfaces and of complex differentiable functions, we will derive Weierstrass' representation formula. It encodes the local differential geometric information of a minimal surface in terms of a pair of analytic functions.

Concrete applications of Weierstrass' formula blossomed in the early 1980's, when David Hoffman realized that computer graphics would allow one to actually visualize the minimal surfaces corresponding to particular Weierstrass data. We will compare some computer creations to what we can make with a bucket of soapy water and wire frames.

What is a minimal surface?

Definition 1 Every point on the surface has a neighborhood which is *the* surface of least area with respect to its boundary

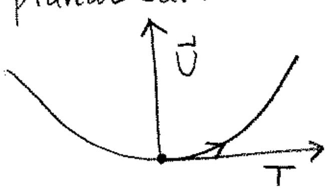
this is why you can see (pieces of) minimal surfaces using wire frames & soap films.

examples: Catenoid – (Euler 1744) (Leibniz, Huygens, Bernoulli 1691)
Helicoid – (Meusnier, 1770's)
Enneper's surface – (?)
flat planes

(equivalent)

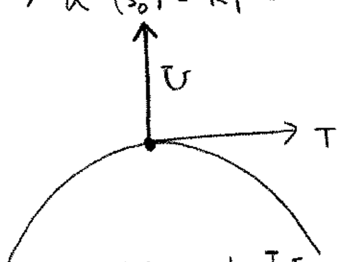
Definition 2 The mean curvature $H = \frac{1}{2}(k_1 + k_2)$ of the surface is identically zero

planar curvature

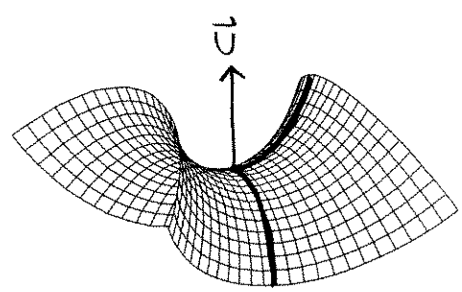


$\vec{\alpha}(s)$ parameterized at unit speed $(\alpha' \cdot \alpha' \equiv 1)$

$\Rightarrow \alpha''(s_0) = k_1 U$



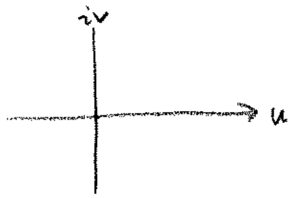
$\beta''(s_0) = k_2 U$



α, β lie in $\perp$ normal plane sections with most bending.

What is a complex analytic function?

$\mathbb{C}$ complex plane: (looks a lot like $\mathbb{R}^2$)



Definition 1

$f: \mathbb{C} \to \mathbb{C}$ is complex analytic at $\vec{z}$

iff $\displaystyle\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} := f'(z)$ exists

- def looks just like in Calculus, for $f: \mathbb{R} \to \mathbb{R}$, but consequences are <u>wildly</u> different! (h is a complex number)
- examples

$$f(z) = z^n$$
$$f(z) = p(z) \quad \text{(polynomials)}$$
$$f(z) = e^z,\ \sin z,\ \cos z, \ldots$$

Definition 2

if $z = u + iv$

$$f'(z) = \frac{\partial f}{\partial u} = -i\frac{\partial f}{\partial v}$$

so if $f(z) = x + iy$

$$\frac{\partial x}{\partial u} + i\frac{\partial y}{\partial u} = -i\left(\frac{\partial x}{\partial v} + i\frac{\partial y}{\partial v}\right)$$

$$\boxed{\begin{aligned} \frac{\partial x}{\partial u} &= \frac{\partial y}{\partial v} \\ \frac{\partial y}{\partial u} &= -\frac{\partial x}{\partial v} \end{aligned}}$$

Cauchy-Riemann equations.

MYSTERY:

How in the world could the concepts of "complex analytic fcns" and "minimal surfaces" be related?

HISTORY!

"When a mathematician understands or discovers something new, the way in which he or she understood things before is rapidly obliterated. Mathematics devours its own history in the process of creating itself"

David Hoffman

Complex Analysis

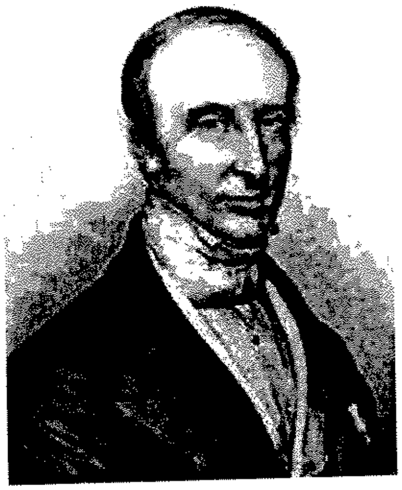
Cauchy Integral formula, etc.

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

1825

Augustin-Louis Cauchy

1789-1857



but earlier came

Cauchy-Riemann Eqtns

$$f(u+iv) = x+iy$$

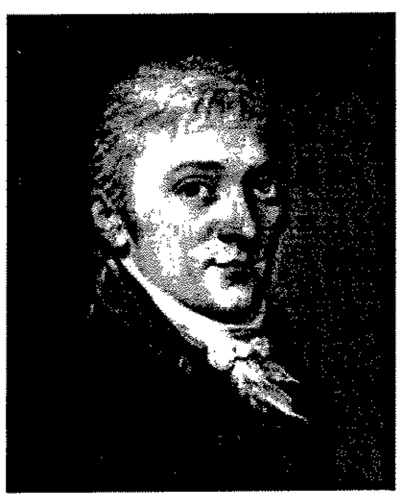
$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}$$

$$\frac{\partial y}{\partial u} = -\frac{\partial x}{\partial v}$$

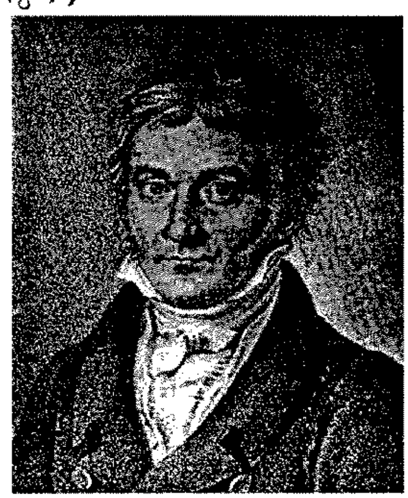
e.g. Leonhard Euler 1777
(d'Alembert 1752)

Carl Friedrich Gauss

1777-1855



In 1803



In 1828

1822

Relates to a "map" question studied by Gauss.

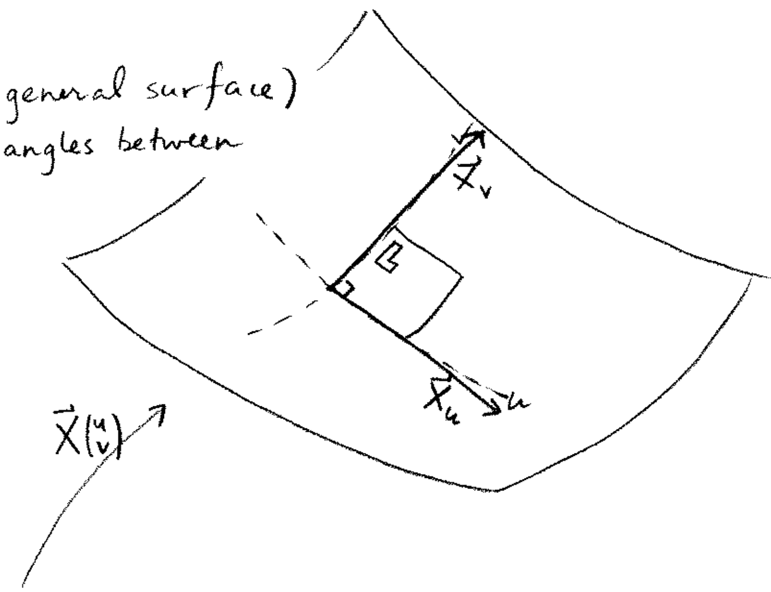
What makes a good map?

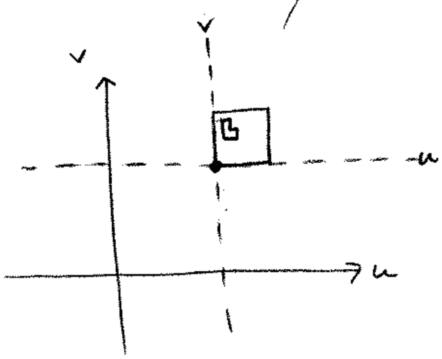
In 1818, at age 31, Gauss contracted to undertake a geodetic survey, for the German state of Hanover, in order to link up with the existing Danish grid. This took (?) years. (He continued to publish math.) To help with his surveying, Gauss invented the <u>heliotrope</u>: "an instrument used in geodetic surveying for making long distance observations by means of the sun's rays thrown from a mirror"

A good map (for a given, general surface) should accurately reflect angles between intersecting curves.

⇕

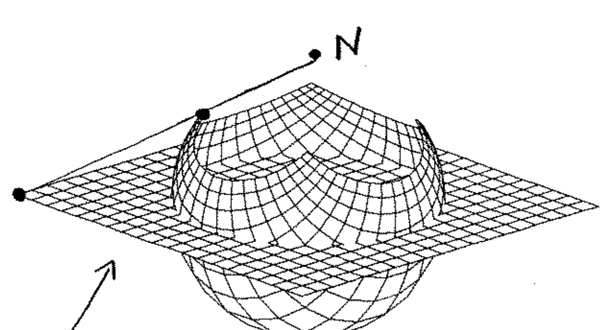
infinitesimal squares are mapped by X to infinitesimal squares on surface





$$\boxed{\begin{array}{c} X_u \cdot X_u = X_v \cdot X_v \\ X_u \cdot X_v = 0 \end{array}}$$

Gauss called such a "map" <u>con</u> <u>formal</u>



(to see Antarctica you look from bottom)

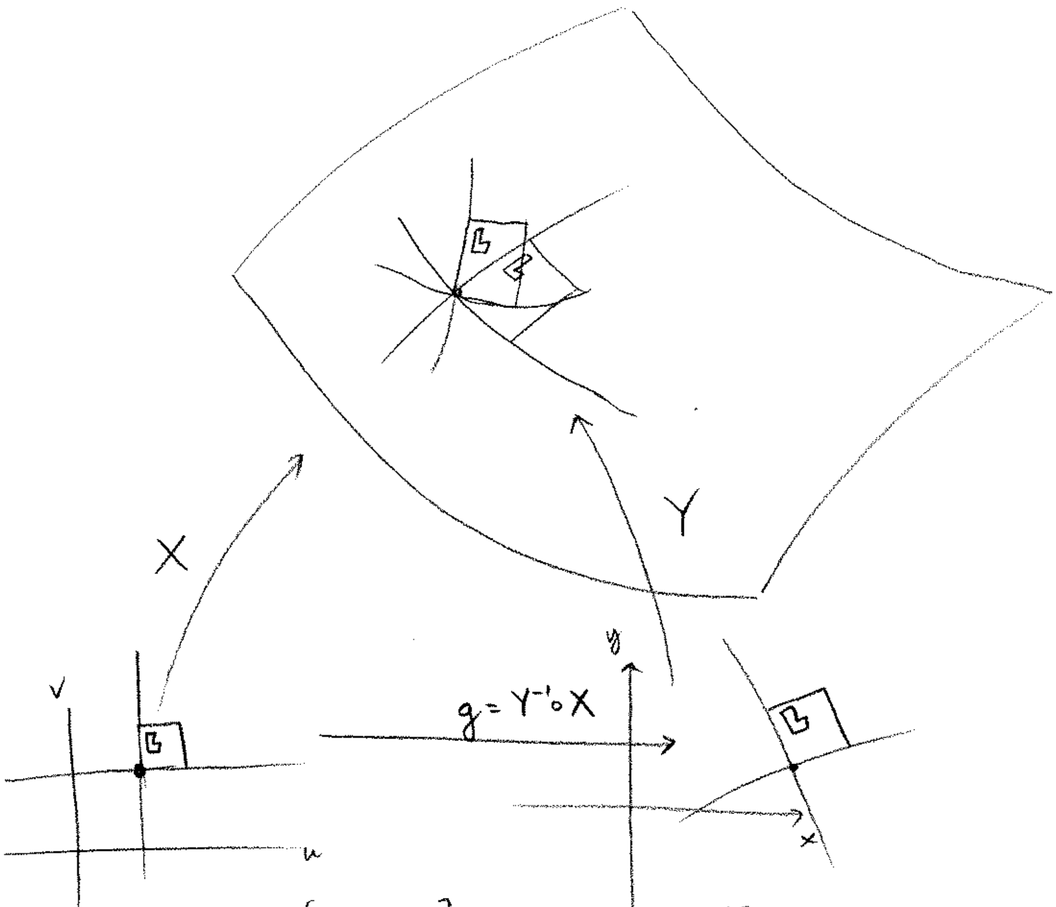
Stereographic projection maps are conformal

Gauss' maps cont'd

In 1822, Gauss received the Copenhagen University prize, partly for showing that one can map one surface onto another so that the two are "similar in their smallest parts" (i.e. conformal)

[nowadays this is sometimes called the "existence of isothermal coords" – we won't prove it here]

But,
How are two (different) good maps of the same surface related?



$$g\left[\begin{pmatrix}u\\v\end{pmatrix}+\begin{pmatrix}h\\k\end{pmatrix}\right] = g\begin{pmatrix}u\\v\end{pmatrix} + \underbrace{\left[\vec{g}_u \middle| \vec{g}_v\right]}_{} \begin{bmatrix}h\\k\end{bmatrix} + \varepsilon$$

rotation dilation matrix iff conformal and orientation preserving

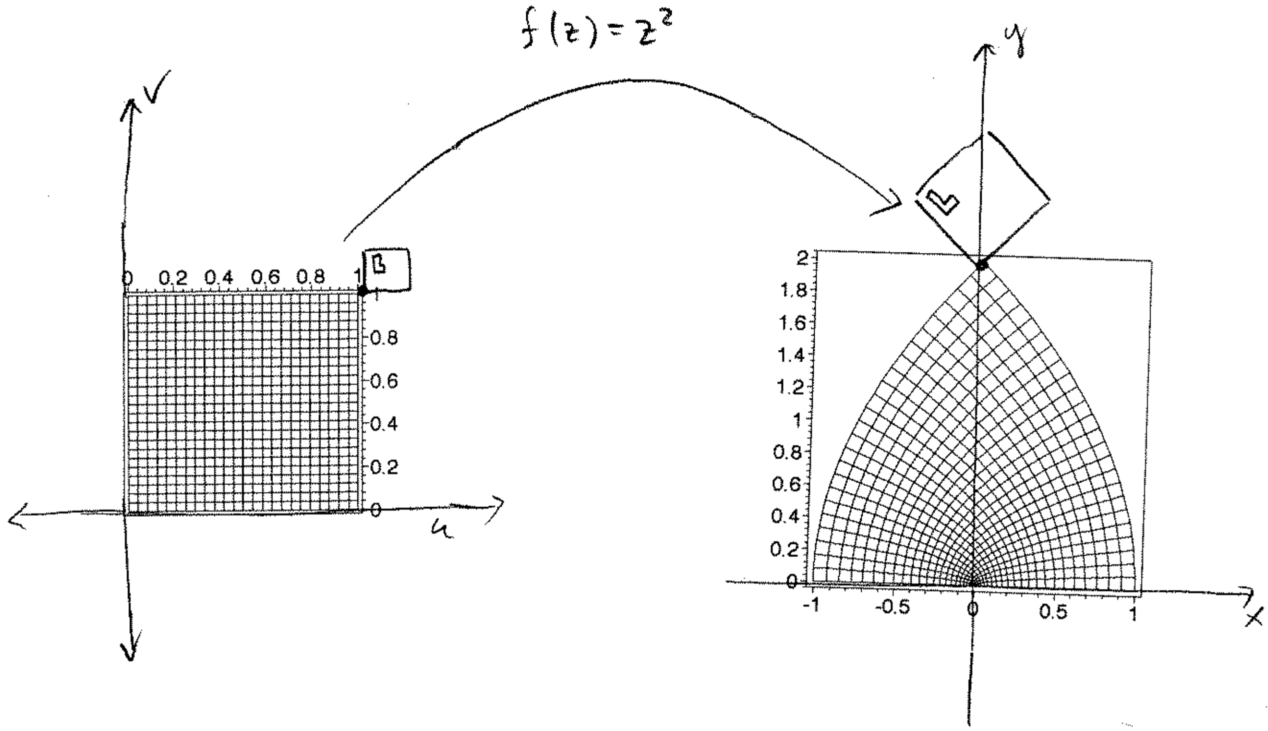
$$\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{"} \sqrt{a^2+b^2}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Cauchy-Riemann!

Def 3

$g: \mathbb{C} \to \mathbb{C}$ is analytic iff g is conformal and orientation preserving

Example



$$f'(z) = 2z$$

$$f\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} u^2 - v^2 \\ 2uv \end{bmatrix}$$

$$f(1+i) = (1+i)^2 = 2i$$

$$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$f'(1+i) = 2(1+i) = \underbrace{2\sqrt{2}}_{\text{dilation}}\, e^{i \underbrace{\pi/4}_{\text{rotation}}}$$

$$Df\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}$$

$$Df\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} = 2\sqrt{2} \begin{bmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{bmatrix}$$

✓ ✓

$$f(z+h) = f(z) + f'(z)h + \varepsilon$$

Applying map theory to minimal surfaces I

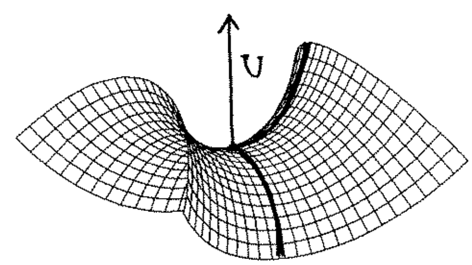
Let $X\binom{u}{v}$ be an isothermal map for the minimal surface M

$$\left.\begin{array}{l} X_u \cdot X_u = X_v \cdot X_v \\ X_u \cdot X_v = 0 \end{array}\right\} \text{isoth.}$$

Def. 3

$$X_{uu} + X_{vv} = 0 \;\Big\}\; \text{Min.}$$

(related to $k_1 + k_2 = 0$)



$$\Updownarrow$$

$\Phi := X_u - i X_v$

satisfies

$\Phi \cdot \Phi = 0$

Φ analytic (in each component)

Def. 4

$(X_u)_u = -(X_v)_v$

$(X_u)_v = -(-X_v)_u$

Enneper's Surface

$$X\binom{u}{v} = \begin{bmatrix} u - \frac{1}{3}u^3 + uv^2 \\ -v - u^2 v + \frac{1}{3}v^3 \\ u^2 - v^2 \end{bmatrix}$$

$$X_u = \begin{bmatrix} 1 - u^2 + v^2 \\ -2uv \\ 2u \end{bmatrix}$$

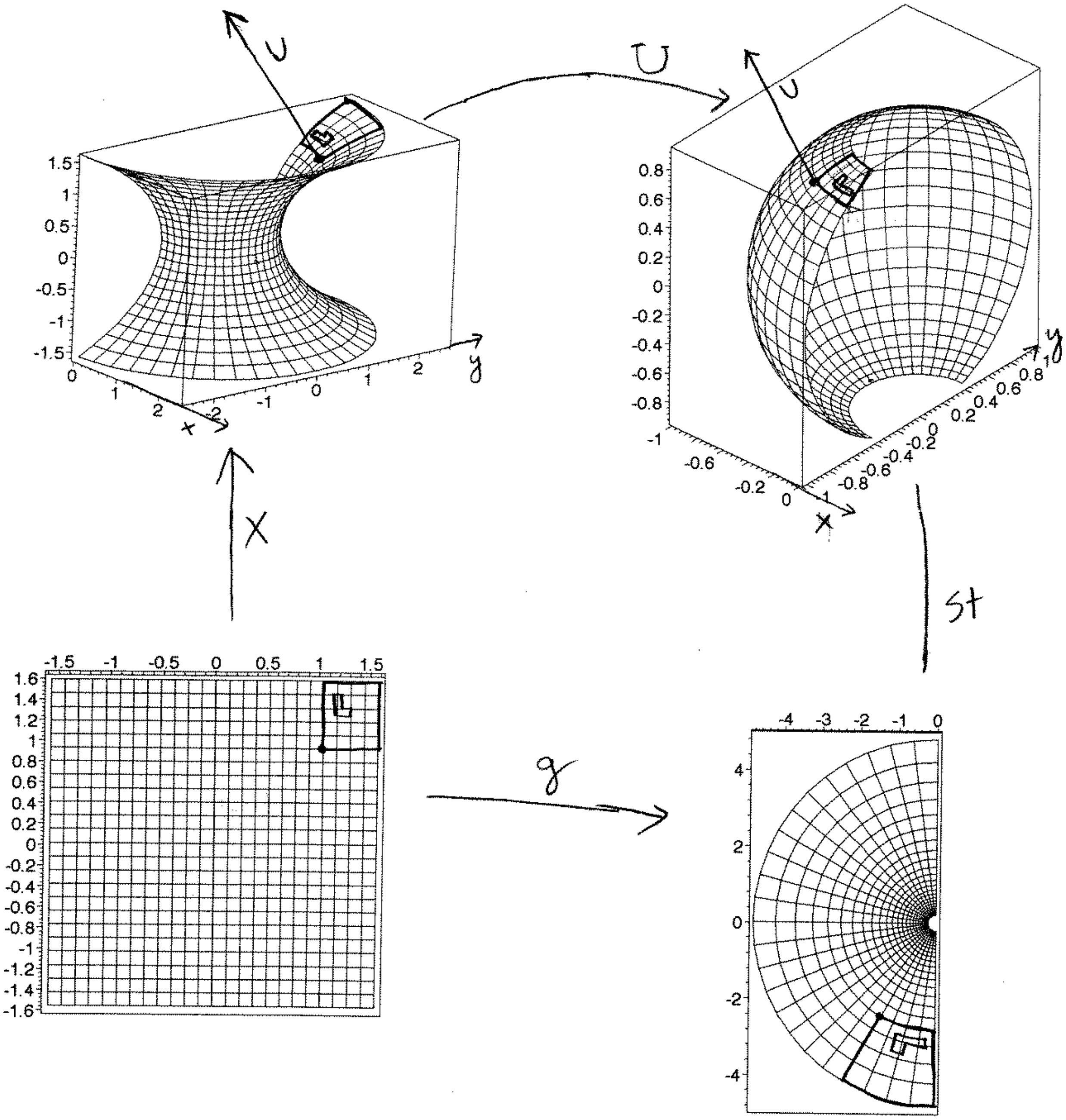
$$X_v = \begin{bmatrix} 2uv \\ -1 - u^2 + v^2 \\ -2v \end{bmatrix}$$

$$\Phi = X_u - i X_v = \begin{bmatrix} 1 - z^2 \\ i(1+z^2) \\ 2z \end{bmatrix}$$

Applying map theory to minimal surfaces II

" the Gauss map U is conformal

Δ $\boxed{St \circ U \circ X = g \text{ is analytic}}$ $\boxed{\text{Def } S}$



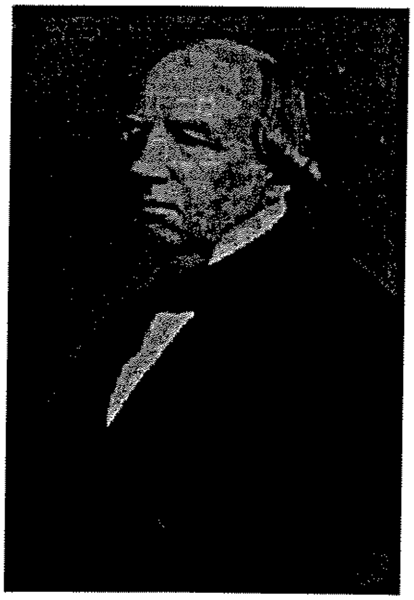
$$X\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \cosh u \cos v \\ \cosh u \sin v \\ u \end{bmatrix}$$

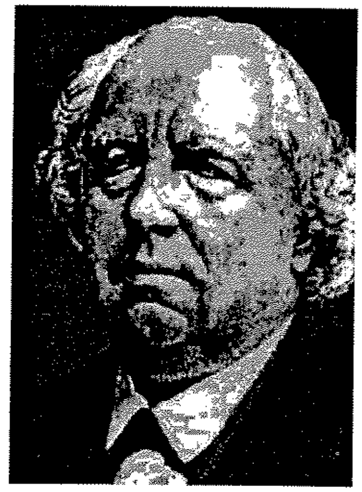
$g = ?$

See Maple!

Karl Weierstrass

1815 – 1897



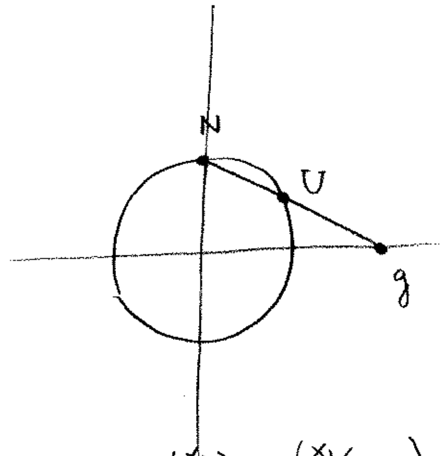


Quote: "It is true that a mathematician who is not also something of a poet will never be a perfect mathematician"

In 1866, Karl Weierstrass figured out how to reverse the previous process:

1. Given analytic g, find a minimal surface with this g
2. Find all minimal surfaces with this g.

①



The map $X\binom{u}{v}$ for which

$$\Phi = X_u - iX_v = \begin{bmatrix} 1-g^2 \\ i(1+g^2) \\ 2g \end{bmatrix}$$

has

$$St \circ U \circ X = g.$$

$$St\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1/1-x_3 \\ x_2/1-x_3 \\ 0 \end{pmatrix}$$

$$St^{-1}\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2a}{1+a^2+b^2} \\ \frac{2b}{1+a^2+b^2} \\ \frac{a^2+b^2-1}{1+a^2+b^2} \end{pmatrix}$$

$\Phi \cdot \Phi = 0$ ✓

Φ analytic ✓

$X_u, X_v \perp St^{-1}(g)$

You get X by integrating Φ

$$X\binom{u}{v} = p + \int_{\binom{u_0}{v_0}}^{\binom{u}{v}} X_u\,du + X_v\,dv = p + Re \int \underbrace{(X_u + iX_v)}_{\Phi(z)}\underbrace{(du + idv)}_{dz}$$

② What other conformal $Y\binom{u}{v}$ have same g?

same U

same Tang plane

$$Y_u = aX_u + bX_v$$
$$Y_v = -bX_u + aX_v$$

$$Y_u - iY_v = f\,(X_u - iX_v)$$

integrability: $Y_{uv} = Y_{vu}$:

$$\boxed{a_v X_u} + a X_{uv} + \boxed{b_v X_v} + \cancel{b X_{vv}}$$
$$= \boxed{-b_u X_u} - \cancel{b X_{uu}} + \boxed{a_u X_v} + a X_{vu}$$

$a_u = b_v$

$a_v = -b_u$

$f := a + bi$ analytic!

$\boxed{\text{Weierstrass - Enneper representation for pair } (f, g) \text{ of analytic fns}}$

$$\Phi := Y_u - i Y_v = \begin{bmatrix} f(1-g^2) \\ i f(1+g^2) \\ 2fg \end{bmatrix}$$

$g = St \circ \nu \circ Y$
so determines ν & tangent plane

f rotates & dilates the tangent plane

$$Y = \int_{\text{path}} Y_u \, du + Y_v \, dv \; = \mathrm{Re} \int \Phi(z) \, dz$$

see Maple
& Netscape.

<u>example</u>

$(f, g) = (1, z)$

$$\Phi = \begin{bmatrix} 1 - z^2 \\ i(1+z^2) \\ 2z \end{bmatrix} \qquad \int \Phi \, dz = \begin{bmatrix} z - z^3/3 \\ i(z + z^3/3) \\ z^2 \end{bmatrix}$$

$$Y = \mathrm{Re}(\%) = \begin{bmatrix} u - \frac{1}{3}u^3 + uv^2 \\ -v - u^2 v + \frac{1}{3}v^3 \\ u^2 - v^2 \end{bmatrix}$$

<u>Enneper!</u>