Mathematics 2270-2280
Linear Algebra-Introduction to Differential Equations
Sequence Overview and Instructor Notes

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1 Sequence overview

Mathematics 2270-2280 is a year-long sequence devoted to linear mathematics. The first semester is a course about linear algebra and the second semester is an introduction to ordinary and partial differential equations. Most students taking this course are math majors or minors and will take the entire sequence. Strong engineering and science students are also encouraged to take 2270-2280 as a more complete alternative to the engineering math courses 2250, 3150. Since you should keep both semesters of the sequence in mind even if you are teaching only one of them, these notes cover the entire year.

Students in these two courses are expected to learn the theoretical framework of the mathematics being discussed as well as the practical computational methods which result from the theory. In particular, students should be required to learn key definitions and proofs, especially in the linear algebra course. The text by Bretscher includes true-false questions in which students must justify their answers, and you should use these to help guide your class towards conceptual thought. I always include a few theoretical questions on my exams, sort of warmups for what students will be confronted with in higher-level math courses. For example, can a student prove that the general solution to an inhomogeneous linear equation is given by the sum of a particular solution with the general solution to the homogeneous equation? Can they define a linear operator? Can they derive the general solution to a first order linear differential equation and explain what its form has to do with the previous linear algebra problems? Of course, I give students some indication of the definitions and theorems they are responsible for before I test them.

Computationally, it is expected that students will become proficient in using Maple (or equivalent) software on the computer in order to complete large or complicated computations, and for visualization. If you believe strongly in Matlab vs. Maple, you are free to teach your students via that particular package instead. It is not assumed that Math 2270 students have previous Maple/Matlab experience, and the first project should partly be an introduction to the software being used. It is not an option to omit computer projects from the coursework. I have taught Math 2270 and Math 2280 several times, and encourage you to use my old projects or modify them as you please. You can find them by following links from my home page. If you choose to expand upon my efforts or to create your own, please let me know about your successful ideas so that we may maintain an on-line collection of links.

2 Texts

- Math 2270: *Linear Algebra with Applications 3E*, by O. Bretscher; ISBN=9780131453340
- Math 2280: *Differential Equations and Boundary Value Problems: Computing and Modeling, fourth edition*, by C.H. Edwards and D.E. Penney; ISBN=9780131561076

3 Prerequisites

For Math 2270, students need to have succeeded at first-year Calculus: Mathematics 1210-1220 (or 1250-1260, or 1270-1280). Math 2270 is a prerequisite for Math 2280. The 2280 students would also benfit from the multivariable calculus in either 1260 or 2210; they need an understanding of curves and tangent vectors to understand the geometric meaning of solutions to systems of differential equations, and they should understand partial derivatives and the chain rule to understand linearization (especially near equilibria of non-linear systems), and to make sense of partial differential equations.

4 Grading

In each course there should be at least two in-class midterms as well as a comprehensive final exam. Exams should be graded primarily by the instructor. Homework should be assigned regularly. The assigned homework should be collected and (at least partially) graded, or there should be frequent short quizzes on course and homework material. It is important that students get frequent, timely feedback on their work. Contact Angie Gardiner (585-9478, gardiner@math.utah.edu) to request a grader for your section. Do this ASAP, preferably before the semester begins.

5 Computer projects

The Department strongly suggests that you include computer work in both Math 2270 and Math 2280. Software output enables you to discuss interesting examples in class, which would be too difficult to work by hand. As well, the computer is great for expressing and visualizing quantitative and qualitative behavior. Your students may not know whether their interests are primarily pure or applied, and these two gateway courses to the major should include elements which are theoretical, as well as ones which are applied. My preference is to use MAPLE, although instructors in these classes have also used MATLAB successfully. Hopefully by the end of this sequence students will feel comfortably using math software to tackle computationally challenging problems, or even as an aid to understanding theoretical concepts. As the year progresses I often include homework problems which have a mathematical software component, in addition to the larger "projects".

For the first couple of projects in Math 2270 I recommend meeting with your class in the computer lab LCB 115, as there is no assumption that your students have any previous experience with Maple or Matlab. Contact Angie Gardiner, before the semester starts if possible, to reserve the LCB 115 computer lab for the By the end of 2280 I simply post the projects and let the students work on them outside of class. I do often hold some of my office hours in the Rushing Student Center lab, when a project due date is approaching.

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7 Math 2270 details

Covering the first 8 chapters of the text is definitely possible, although a bit rushed. The Spectral Theorem for symmetric matrices, and singular value decomposition (Chapter 8) are very important concepts, so if at all possible complete this chapter. This text is excellent at integrating geometric and algebraic concepts. It is written at a level of sophistication that is above what students coming out of Calculus are used to, however, so make sure to slow down and fill in details at difficult conceptual junctures.

The semester begins with the basic study of linear equations, the algebra and geometry of linear transformations, matrix and transformation inverses, in Euclidean space.

I have a project on fractals via interated function systems which I have taught in my own class at this point of the course. It is a fun way for the students to solidify the geometric meaning of affine maps and to introduce them to Maple. Go to my directory korevaar/fractals to see what this module is about. This method of fractal generation has an interesting theoretical backing based on the contraction mapping principle from analysis, and on Hausdorff distance between compact sets. I would be happy to provide you with material for this module or even to present it in your class, if you ask.

Next, Bretscher very quickly introduces subspaces, span, independence, basis, and dimension, in the context of Euclidean space, before moving to abstract vector spaces. This material will be used extensively in the 2280 semester.

Orthogonality in Euclidean space is discussed, including Gram-Schmidt orthogonalization, projection, and the the method s of least squares for finding optimal approximate solutions to overdetermined systems as well as to data fitting. The idea of of Euclidean orthogonality generalizes to inner product spaces, such as the L^2 function spaces. I mention Fourier series in this context so that they seem a little less mysterious in Math 2280.

Bretscher holds off until chapter 6 to explain determinants, and includes a good section on their geometric meaning. I would mention the multivariable change of variables formula in integral calculus in this context; it is not treated very completely in our multivariable calculus course. Chapter 7 is devoted to eigenvalues and eigenvectors. The material is motivated with interesting discrete dynamical systems. This is a good time to review complex numbers and complex linear algebra, as does the text. If you have time and an energetic class it could also be appropriate to talk about Jordan canonical form for non-diagonalizable matrices, although I wouldn't try to include proofs of everything. In chapter 8 the text uses eigenvector analysis to prove the spectral theorem about diagonalizing symmetric matrices, and applies this result to quadratic forms, and hence to conics and quartic surfaces. I use this opportunity to talk about the multivariable second derivative test as well. The final section explains the singular value decomposition of a linear transformation between Euclidean spaces. This decomposition has many applications in numerical analysis.

8 2270 suggested lectures

The following estimates add up to 48 lectures. In the fall (resp. spring) of the typical academic year there are 58 (resp. 57) MTWF class meetings, leaving time for supplementary topics, Maple labs, exams and reviews (in theory). Feel free to modify these recommendations based on your own inclinations, but don't shortchange the core material. You will find that when Bretscher comes to a complicated theoretical concept (coordinates with respect to a basis comes to mind) his presentation is very concise. This means you will have to add material and examples and time for your class to really understand it.

- Chapter 1: Linear Equations and Matrices - 4 lectures
- Chapter 2: Linear Transformations - 5 lectures
- Chapter 3: Subspaces of Euclidean Space, and dimension - 6 lectures
- Chapter 4: Linear (i.e. vector) Spaces - 5 lectures
- Chapter 5: Orthogonality and Least Squares - 8 lectures
- Chapter 6: Determinants - 5 lectures
- Chapter 7: Eigenvalues and eigenvectors - 9 lectures
- Chapter 8: Symmetric matrices and quadratic forms - 6 lectures.

9 2270 computer projects

These projects will be assigned to enhance the course material. Aim for two or three substantive projects, but you need not use mine. In addition to the projects I expect to work Maple examples into class lectures and regular homework. Project topics I have used in the past:

- Fractal generation by iterated function systems of affine maps - relates to the geometric interpretation of linear transformations, chapter 2. This project also included an introduction to Maple and the Math lab.
- Least square data fitting, and function orthogonality, topics from chapter 5. I like having the students collect height-weight data and using it to derive an empirical power law from the ln-ln data. (They always get a power law for weight as depending on power of height, with power between 2.35 and 2.7, and national data indicates that 2.6 would be good power.
- Conic sections and quadric surfaces, chapter 8.

Mladen Bestvina did an interesting project in spring 2002 with discrete dynamical systems, chapter 7. He's delinked the project, but if you contact him I'm sure he can point you where to go. Let me know if you come up with successful projects so that I can pass your ideas on to future instructors.

10 Math 2280 details

The semester begins with first order differential equations: their origins, geometric meaning (slope fields), analytic and numerical solutions. The logistic equation and various velocity and acceleration models are studied closely. The next topic is linear DE's of higher order, with the principal application being mechanical vibrations (friction, forced oscillations, resonance). At this point we show how various scientific models of dynamical systems lead to first order systems of differential equations, and then we complete the theoretical study of linear systems of DE's which was begun in 2270. The concepts of the phase plane, stability, periodic orbits and dynamical-system chaos are introduced with various ecological and mechanical models. The study of ordinary differential equations concludes with an introduction to the Laplace transform. The final portion of Math 2280 is an introduction to the classical partial differential equations: the heat, wave and Laplace equations, and to the use of Fourier series and separation of variable ideas to solve these equations in special cases. Time permitting, one can also introduce the Fourier transform.

11 2280 suggested lectures

The following estimates add up to 50 lectures. In the typical fall (spring) term there are 58 (resp 57) MTWF class meetings, leaving some time for Maple labs, reviews, and exams. If you find this schedule too tight, you might omit 3.7 and 9.7, for example. Do not short-change chapters 1-6, and try to spend enough time on chapter 9 so that students get some feel for partial differential equations.

- **Chapter 1: First-order differential equations** - 5 lectures
  - 1.1 Differential equations and mathematical models
  - 1.2 Integrals as general and particular solutions
  - 1.3 slope fields and solution curves
  - 1.4 Separable equations and applications
  - 1.5 Linear first-order equations
- **Chapter 2: Mathematical Models and Numerical Methods** - 5 lectures
  - 2.1 Population models
  - 2.2 Equilibrium solutions and stability
  - 2.3 Acceleration-Velocity models (2 lectures)
  - 2.4-2.6 Numerical methods survey
- **Chapter 3: Linear Equations of Higher Order** - 8 lectures
  - 3.1-3.2 Introduction: second-order linear equations, and general solutions of linear equations
  - 3.3 Homogeneous equations with constant coefficients (2 lectures)
  - 3.4 Mechanical vibrations

- 3.5 Nonhomogeneous equations and the method of undetermined coefficients
- 3.6 Forced oscillations and resonance (2 lectures)
- 3.7 Electrical circuits

- **Chapter 4: Introduction to Systems of Differential Equations** - 2 lectures
  - 4.1 First-order systems and applications
  - 4.3 Numerical methods for systems
- **Chapter 5: Linear Systems of Differential Equations** - 7 lectures
  - 5.1-5.2 General theory and the eigenvalue-eigenvector method for homogeneous systems (2 lectures)
  - 5.3 Second-order systems and mechanical vibrations
  - 5.4 Multiple eigenvalue solutions
  - 5.5 Matrix exponentials and linear systems (2 lectures)
  - 5.6 Nonhomogeneous linear systems
- **Chapter 6: Nonlinear Systems and Phenomena** - 6 lectures
  - 6.1-6.2 Stability and the phase plane, and linearization (2 lectures)
  - 6.3 Ecological Applications: Predators and Competitors (2 lectures)
  - 6.4 Nonlinear mechanical systems
  - 6.5 Chaos in dynamical systems
- **Chapter 7: Laplace Transform** - 5 lectures
  - 7.1 Laplace transforms and inverse transforms
  - 7.2 Transformation of initial value problems
  - 7.3 Translation and partial fractions
  - 7.4 Derivatives, integrals, and products of transforms
  - 7.6 Impulses and delta functions
- **Chapter 9: Fourier Series Methods** - 12 lectures
  - 9.1 Periodic functions and trigonometric series (2 lectures)
  - 9.2 General Fourier series and convergence
  - 9.3 Even-odd functions and termwise differentiation (2 lectures)
  - 9.4 Applications of Fourier series to forced oscillation problems
  - 9.5 Heat conduction and separation of variables (2 lectures)
  - 9.6 Vibrating strings and the one-dimensional wave equation (2 lectures)
  - 9.7 Steady-state temperature and Laplace's equation (2 lectures)

12 2280 computer projects:

These projects will be assigned to enhance the course material. Aim for at least 3 substantive projects, in addition to regular homework which has computational aspects. Possible topics for 2280 projects include:

- Slope fields, Euler's method for 1st-order differential equations (Chapters 1-2
- Newton's law of cooling for a house (Chapter 1)
- modeling populations with the logistic equations (Chapter 2)
- modeling springs: damping, forced oscillations, resonance and approximate resonance (Chapter 3)
- Earthquakes and multi-story building vibrations (Chapter 5)
- pplane investigation of non-linear phase portraits (Chapter 6)
- Numerical methods for systems, chaos in Duffing's spring equation (Chapter 6)
- Fourier series (Chapter 9)
- Fourier series solutions to the heat and wave equations (Chapter 9)