

Scaling, Self Similarity & Fractals

Lecture notes for Access 2008

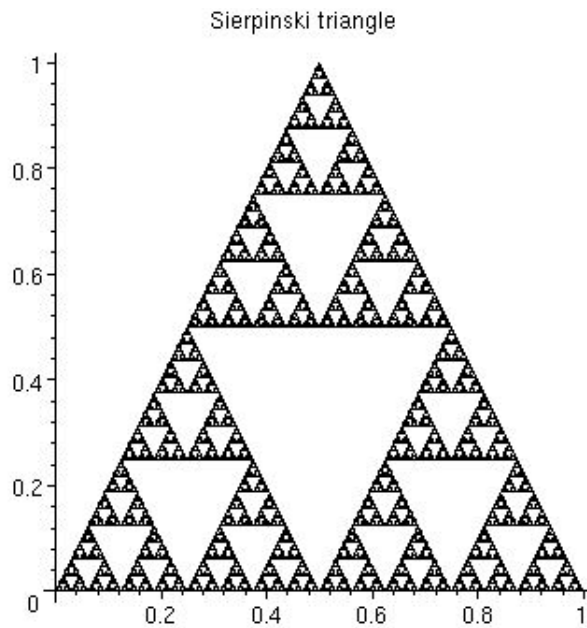


Figure 1: Sierpinski's Triangle

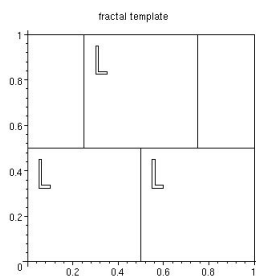
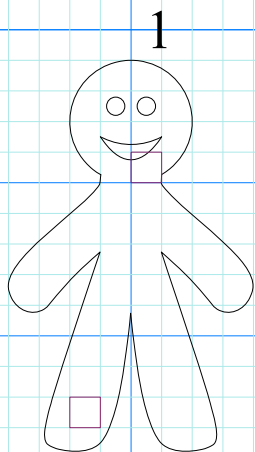
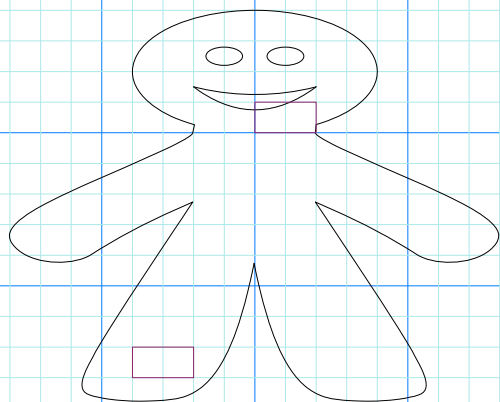
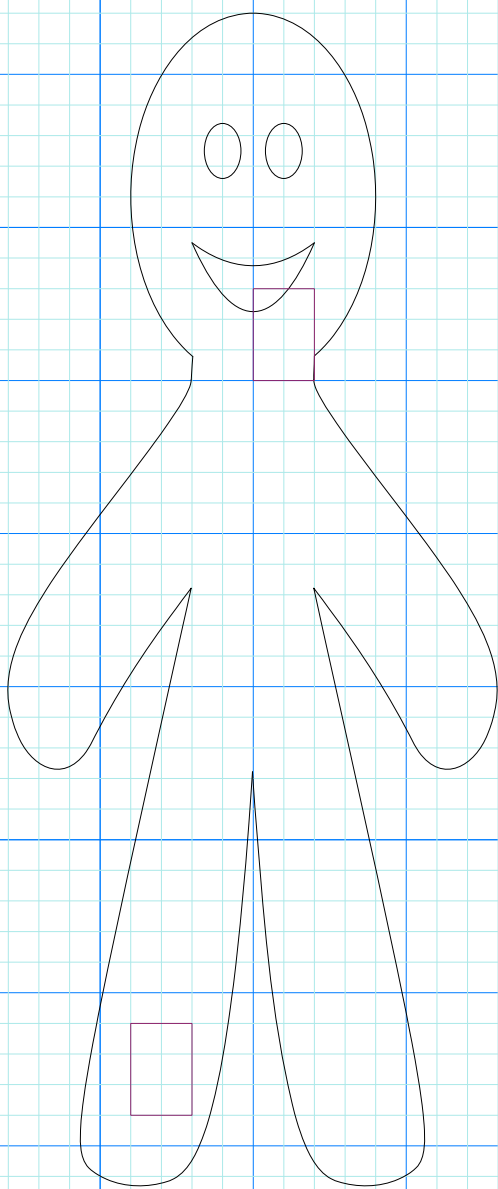
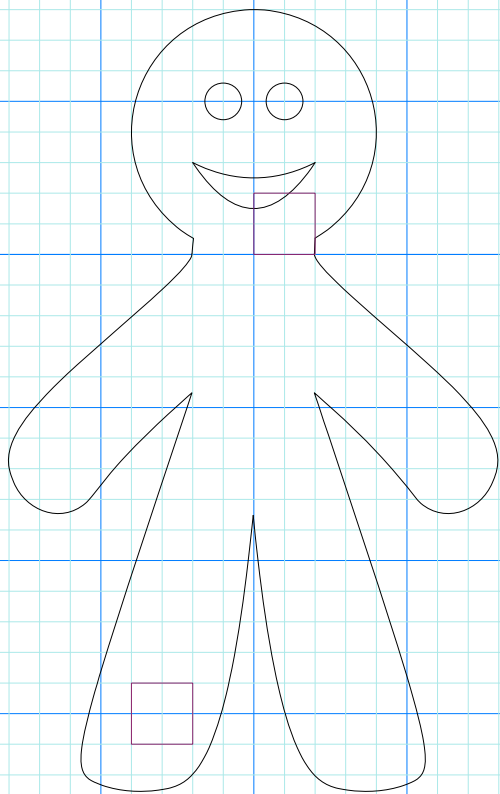
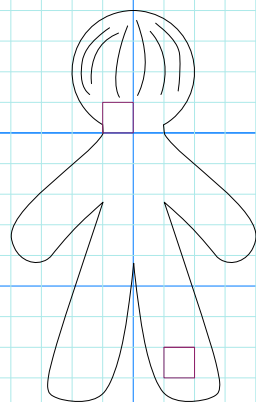
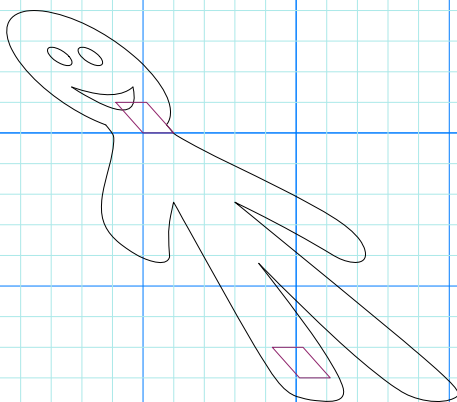
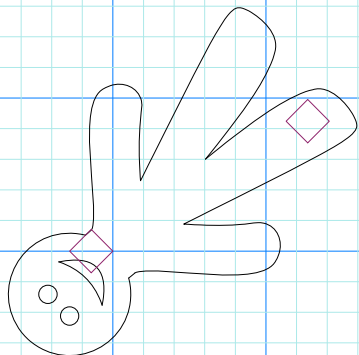
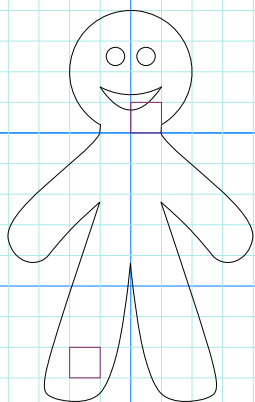


Figure 2: Sierpinski's Construction



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Part I: Classical Scaling

Bob transforms himself. How does he do it?

"Bob" is a collection of points in the plane. If we wish to translate Bob, say 2 units to the right and 1 unit up, we use the transformation function

$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2 \\ y+1 \end{bmatrix} := \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

In the above equation, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is called the translation vector.

We have the following notation: $A(\text{Bob})$ means the set of $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$'s, where $\begin{bmatrix} x \\ y \end{bmatrix}$ is in Bob.

Easy Moves:

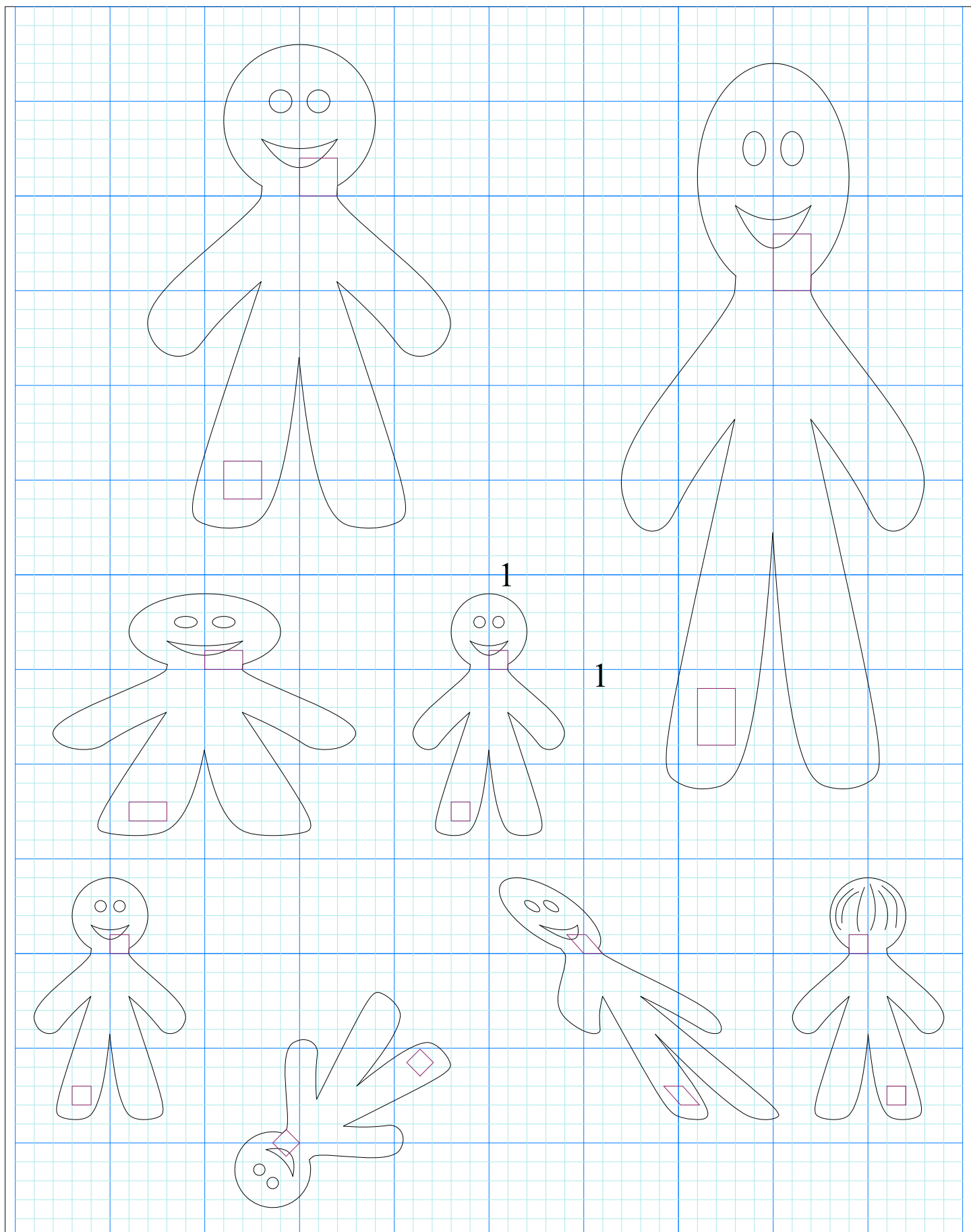
1. Translation: $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$
2. Uniform Scaling, then Translation (say by R): $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} Rx \\ Ry \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$
3. Nonuniform Scaling, then Translation:

Example 1. Find a formula for the transformation which stretches by a factor of a in the horizontal direction, by a factor of d in the vertical direction, then translates.

4. Turning Bob over?

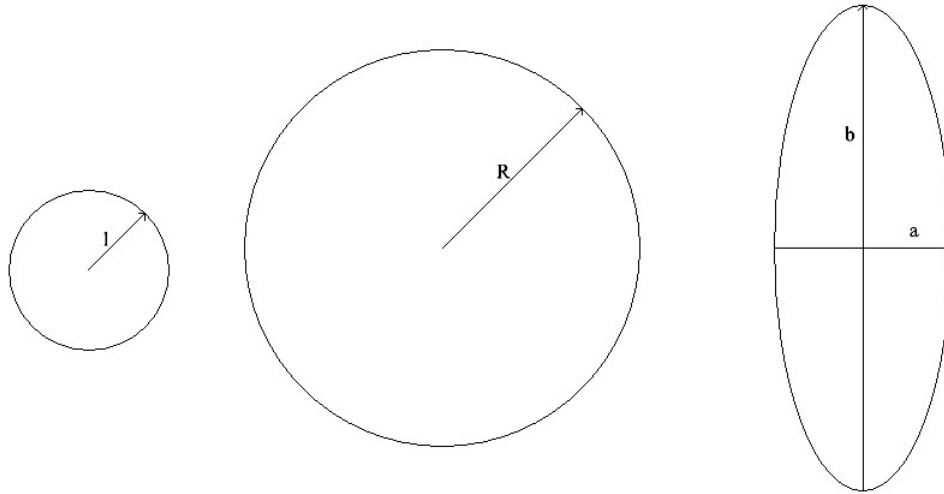
Example 2. Find formulas to turn Bob over, by reflecting him across the x -axis. How do you reflect Bob across the y -axis?

Exercise: Bob transformed himself. What formulas did he use? How did his area change in each case, i.e., how many times larger is the transformed area, compared to the original Bob area? What can you say about the transformed circumferences?



What does scaling do to areas, volumes, and lengths?

Exercise 1. The unit disk has area π and circumference 2π . Using these facts and scaling properties deduce the area of the radius R disc, and the circumference of the radius R circle. What can you say about the area and circumference of the ellipse, created from the unit circle configuration by stretching by a horizontally and b vertically?



Exercise 2. The unit ball has volume $\frac{4}{3}\pi$ and surface area 4π . Use scaling to deduce the volume and surface area of the radius R ball. What can you say about the $a-b-c$ ellipsoid volume and surface area?

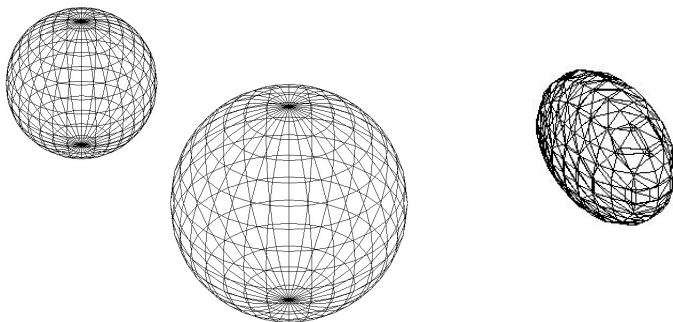
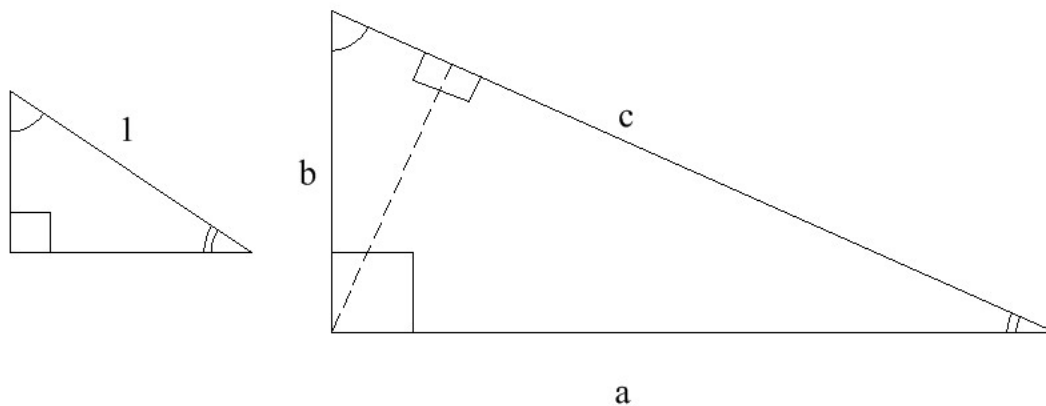


Figure 3: Sphere with radius 1, Sphere with radius R , and a - b - c ellipsoid

Exercise 3. Use similar triangles (there are four of them in the picture below!), and the way area transforms under uniform scaling to find a proof of the Pythagorean Theorem, $a^2 + b^2 = c^2$ for right triangles.



Part II: Harder Transformations

Actually, they are not harder, they are just more general.

$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ax + cy + e \\ bx + dy + f \end{bmatrix}$$

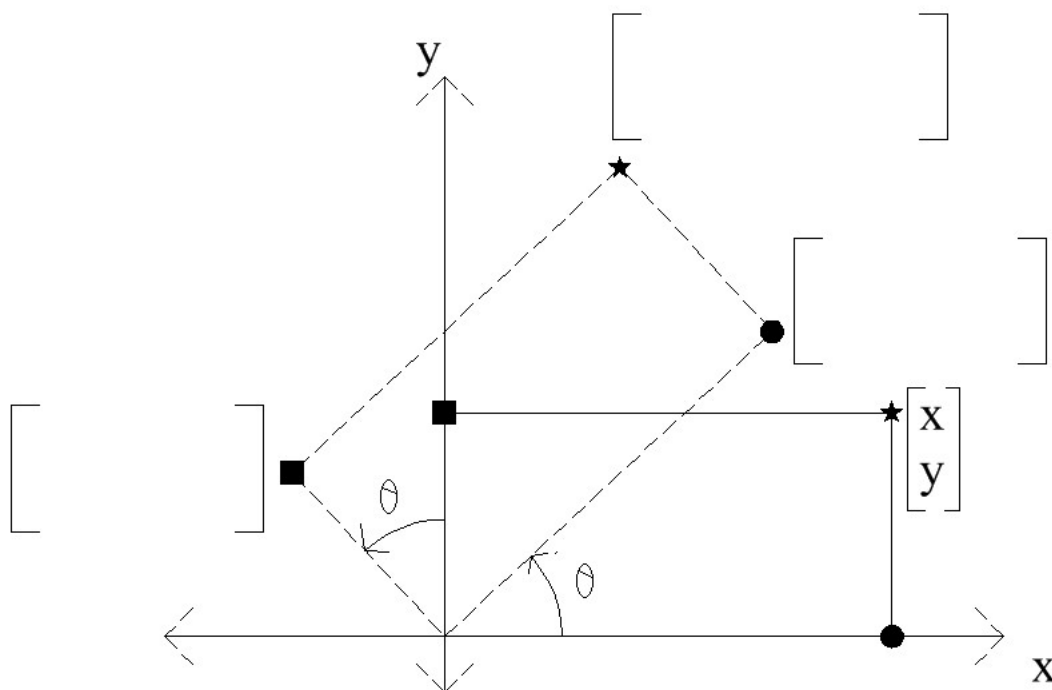
Transformations of this sort are called **affine**. These also include the easy transformations we talked of earlier.

1. Translation: $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \\ \end{bmatrix}$
2. Scaling and Translation: $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 5y \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \\ \end{bmatrix}$
3. Reflection: $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

etc.

There are also rotations. For example we could rotate by θ radians counterclockwise with the following formula:

$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

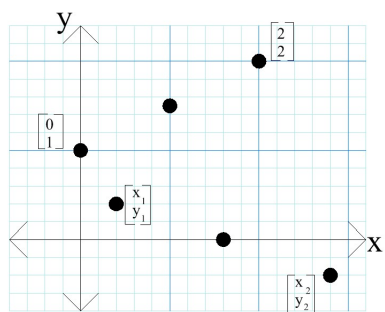


Part III: Geometry of affine transformations

$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

Theorem 1. Let A be an affine map. Then:

1. For any two points P, Q and their midpoint M , the midpoint of $A(P)$ and $A(Q)$ is $A(M)$.



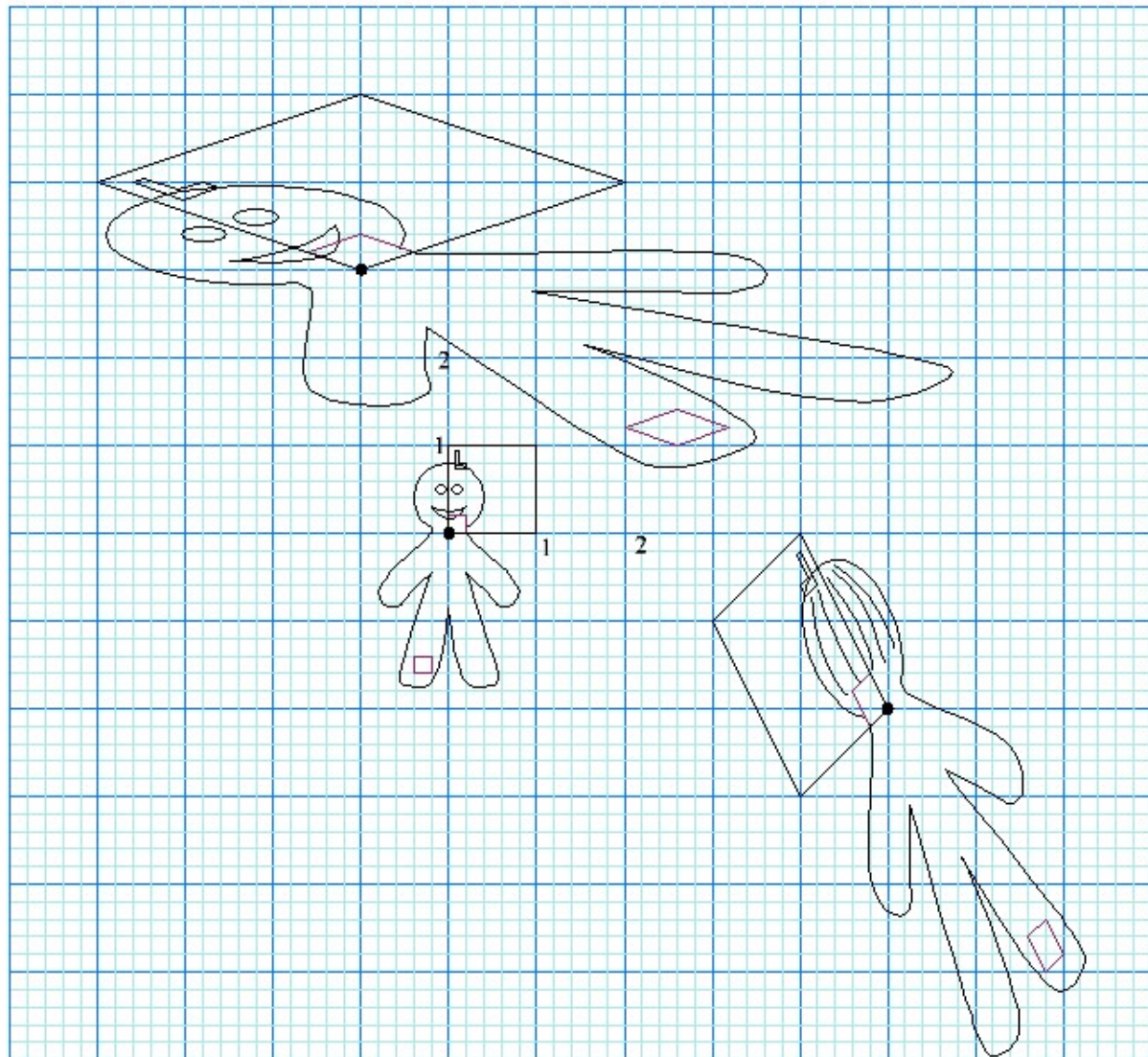
2. Affine maps transform lines into lines.

3. Affine maps transform rectangles into parallelograms, so they also transform rectangular grids into parallelogram grids.

Bob and L-box transform themselves:

Exercise 4. Find formulas for the two affine maps which are shown.

Exercise 5. Show where the L-box is transformed to by $A \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -3 \end{bmatrix}$. If you want, you can draw in a piece of transformed Bob.



Exercise 6. How do general affine transformations affect area? How are the area of “Bob” and the transformed “ $A(\text{Bob})$ ” related? Hints: Since Bob is filled up with squares, and since the area of any square is multiplied by the same area “expansion” (or contraction) factor under an affine transformation, it suffices to find the area of the transformed unit square. Also, translations don’t change area, so you may assume $\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so

$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Find the area of the parallelogram below (in terms of the letters a, b, c, d) to deduce your answer. Is your answer consistent with the earlier exercises in which you computed area expansion factors?

