Scaling, Self Similarity & Fractals

Lecture notes for Access 2007

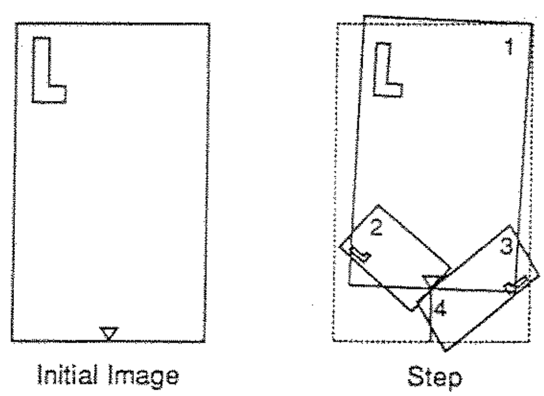


Figure 5.22 : Blueprint of Barnsley's fern.

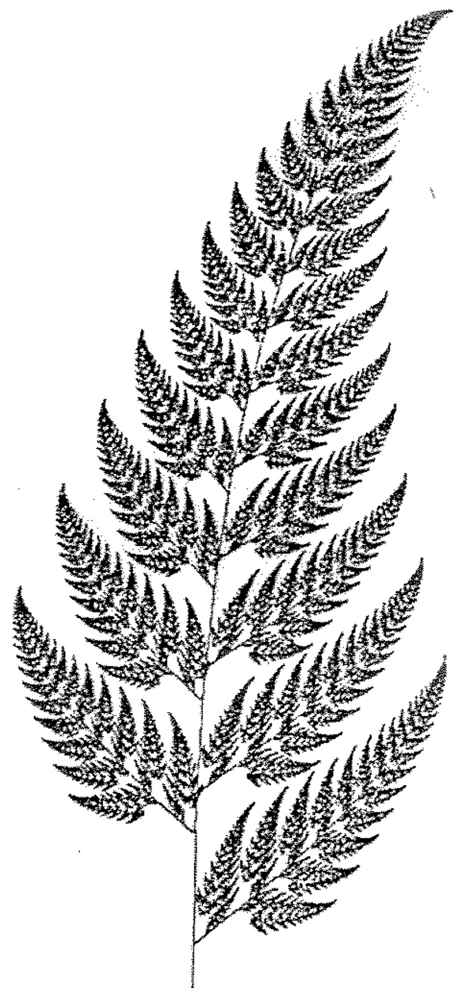
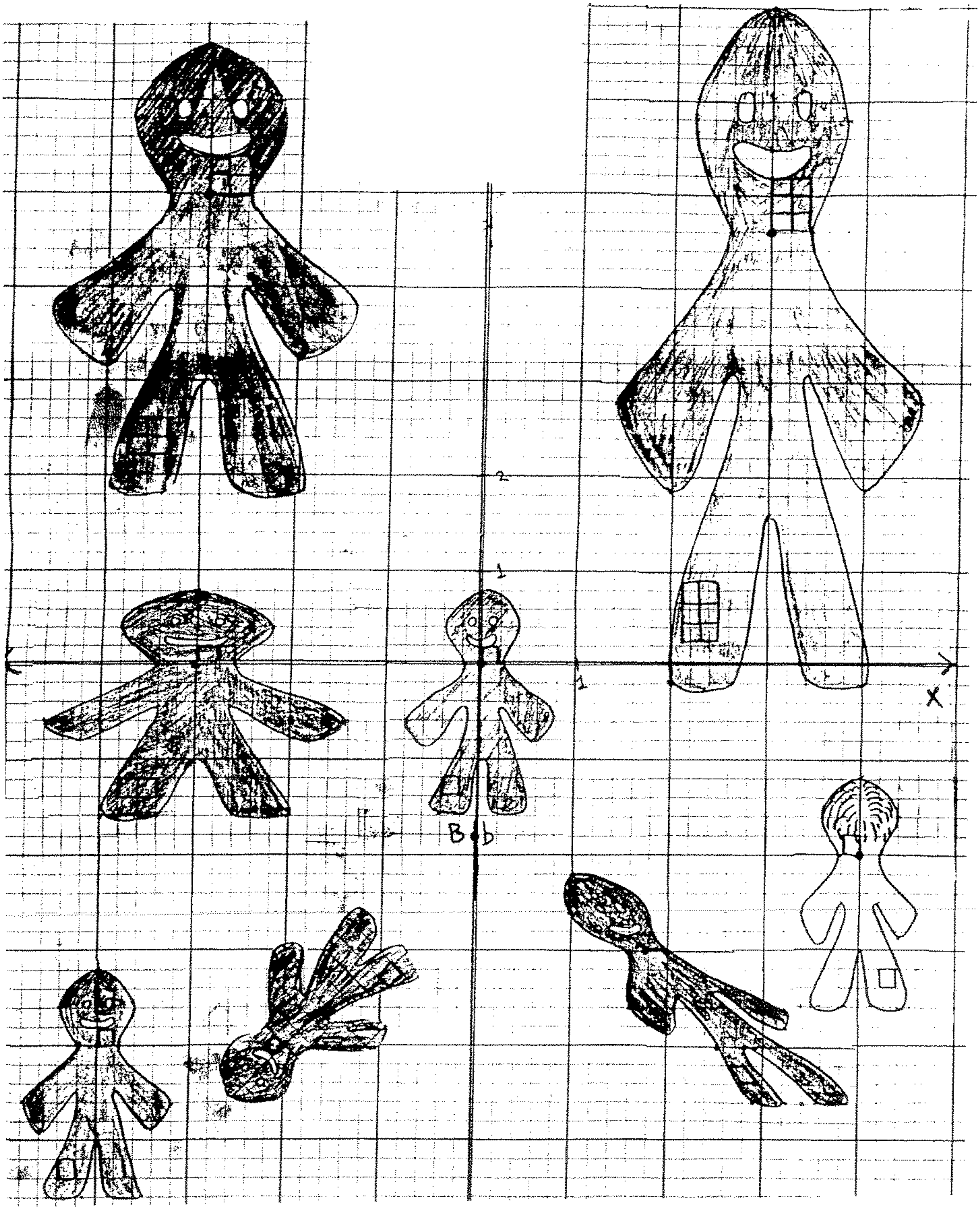


Figure 5.25 : Barnsley's fern generated by an MRCM with only four lens systems.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Part I: Classical Scaling

Bob transforms himself. How does he do it?

"Bob" is a collection of points in the plane. If we wish to translate Bob, say 2 units to the right and 1 unit up, we use the transformation function

$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2 \\ y+1 \end{bmatrix} := \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

In the above equation, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is called the translation vector.

We have the following notation: $A(\text{Bob})$ means the set of $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$'s, where $\begin{bmatrix} x \\ y \end{bmatrix}$ is in Bob.

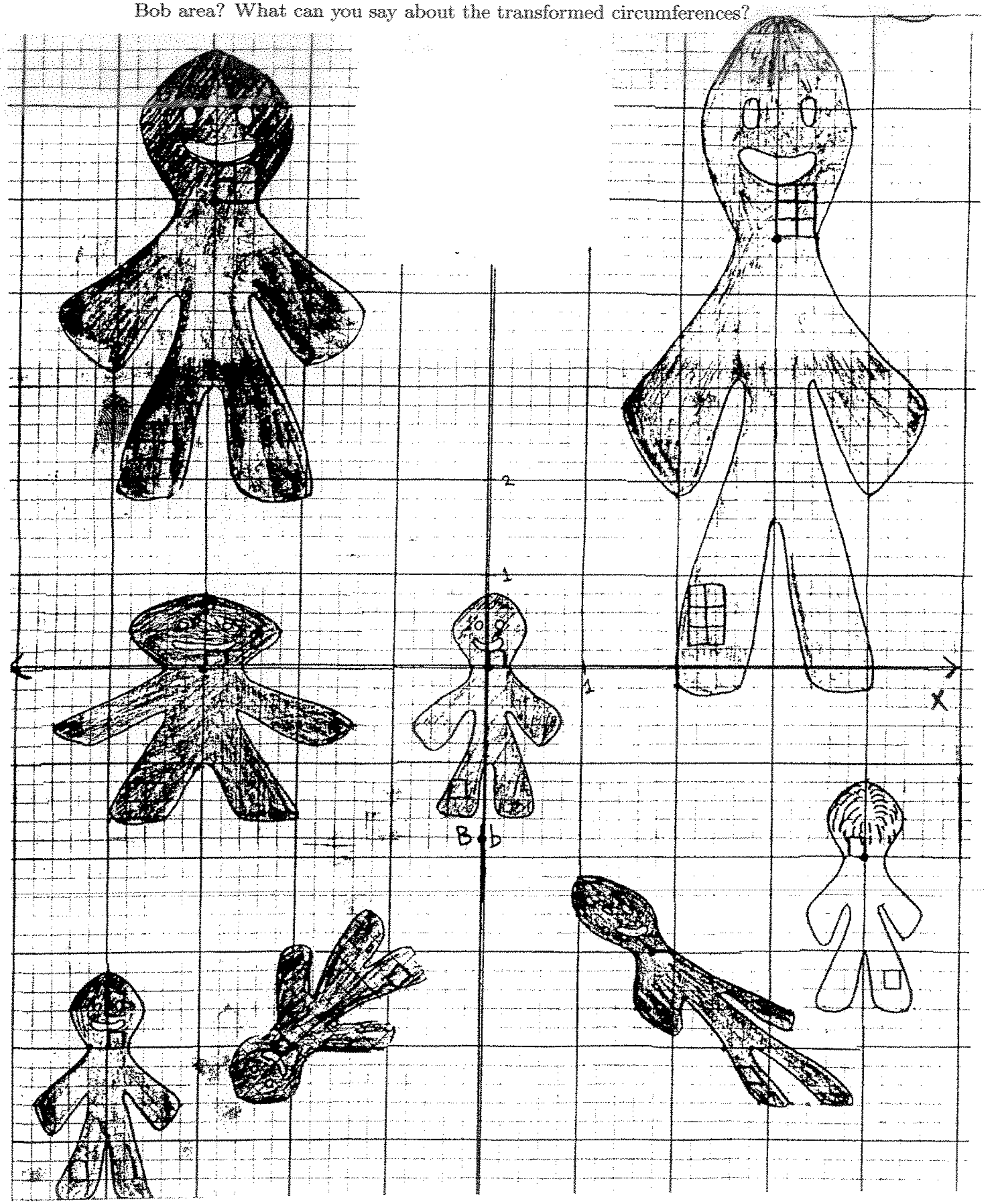
Easy Moves:

1. Translation: $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$
2. Uniform Scaling, then Translation (say by R): $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} Rx \\ Ry \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$

Example 1. Find a formula for the transformation which stretches by a factor of a in the horizontal direction, by a factor of d in the vertical direction, then translates

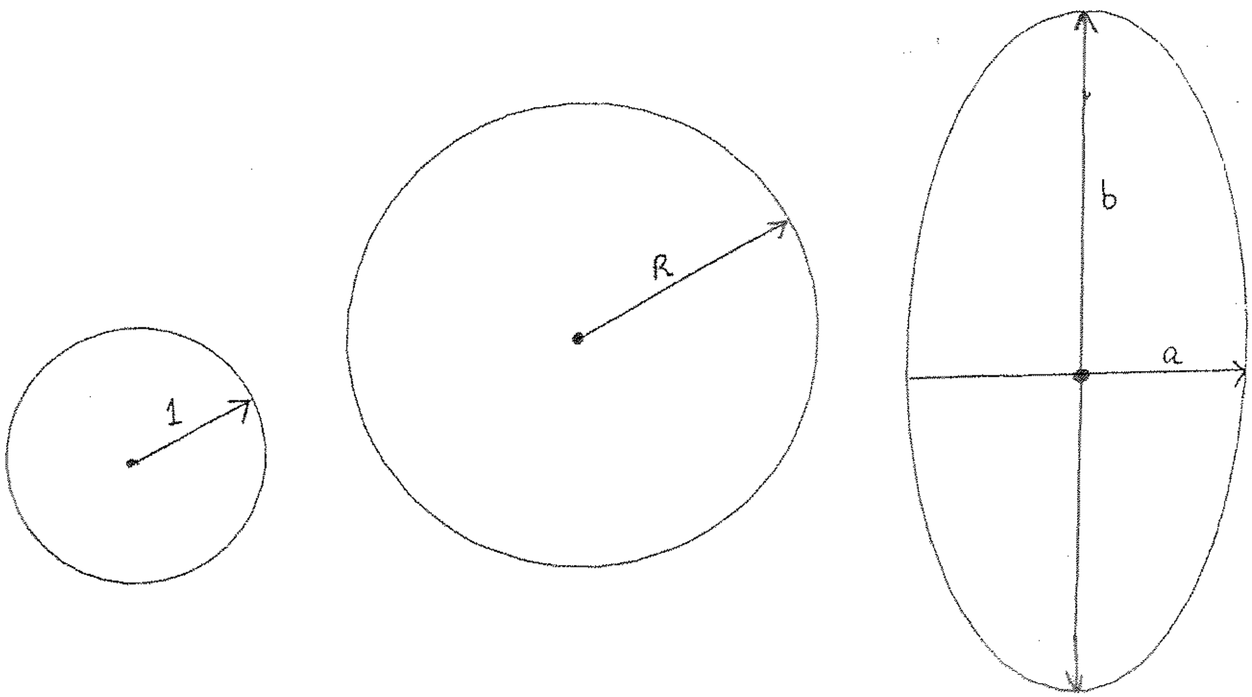
Example 2. Find formulas to turn Bob over, by reflecting him across the x-axis. How do you reflect Bob across the y-axis?

Exercise 1. Bob transformed himself. What formulas did he use? How did his area change in each case, i.e. how many times larger is the transformed area, compared to the original Bob area? What can you say about the transformed circumferences?

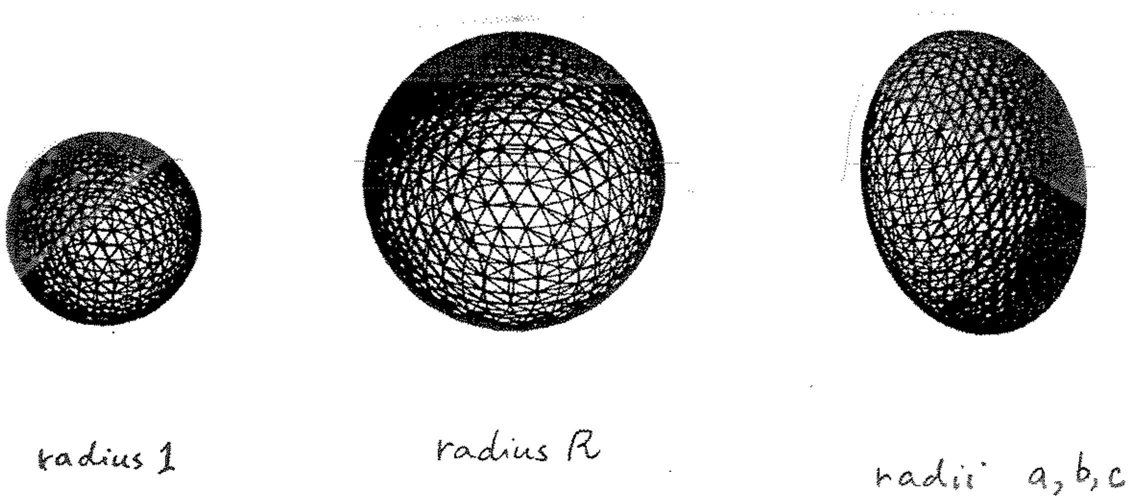


What does scaling do to areas, volumes, and lengths?

Exercise 2. The unit disk has area π and circumference 2π. Using these facts and scaling properties deduce the area of the radius R disk, and the circumference of the radius R circle. What can you say about the area and circumference of of the ellipse, created from the unit circle configuration by stretching by a horizontally, and b vertically?



Exercise 3. The unit ball has volume $\frac{4}{3}\pi$ and surface area 4π. Use scaling to deduce the volume and surface area of the radius R ball. What can you say about the $a-b-c$ ellipsoid volume and area?



Exercise 4. Use similar triangles (there are four of them in the picture below!), and the way area transforms under uniform scaling to find a proof of the Pythagorean Theorem, $a^2 + b^2 = c^2$ for right triangles.