Part IV: Historical Fractals

Georg Ferdinand Ludwig Philipp Cantor
1845(Russia) – 1918(Germany)

Best known for the Cantor set, which is constructed by starting with the unit interval, deleting the (open) middle 1/3 of this interval, and then successively deleting open middle thirds from the remaining set. The points which are never deleted in this process comprise the Cantor Set. A related function is the “Devil’s Staircase” which is continuous and has derivative equal to zero on 100% (but not all!) of it’s domain – yet is not constant. Cantor, who loved set theory and the logical underpinnings of mathematics, suffered from depression and mental illnesses in his later years.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Cantor.html

\[ C_0 \]
\[ C_1 \]
\[ C_2 \]
\[ C_3 \]
\[ C \]

How long is the Cantor Set?
Niels Fabian Helge von Koch
1870-1924 (Sweden)

Best known for the Koch snowflake, made as follows: begin with the unit interval. Delete the middle open third and replace with two intervals of length 1/3, forming two sides of an equilateral triangle. Iterate this process, at each stage replacing each edge with 4 sub-edges having 1/3 the original edge length!

Take the limit of this process (?!), and glue three of the limit pieces together in an equilateral fashion to create the Koch snowflake.

K₀

K₁

K₂

K₃

...

K

How long is K?
Waclaw Sierpinski
1870-1924 (Poland)

You construct the Sierpinski Triangle as follows:
Begin with the a closed equilateral triangular region,
Say with triangle edge lengths equal to 1.
At stage 1 subdivide this triangular region into
four equilateral subregions, with new edges having
half the previous side lengths. Remove the open
central sub-triangular region (keeping the edges).
Continue this process inductively. The Sierpinski
Triangle is the collection of points from the original
closed region which never get deleted.

S
Sierpinski’s Triangle
What is its area?
You can also make fractals in 3 dimensional space....

Sierpinski Pyramid

Karl Menger
1902 (Austria) - 1985 (Chicago USA)

Menger Sponge
(How much volume?)
Scaling (Self-Similarity) Dimension

The concept scaling dimension builds on the ideas we used to understand how area, length, and volume of objects changed under uniform scaling of the background space. There is a much more general notion of dimension, called “Hausdorff dimension”, which is also much harder to explain.

Examples:

(1) If you scale a line segment by an integer factor of $M$, you get $M^1$ pieces which are congruent to the original line segment.
(2) If you rescale a square by an integer factor of $M$, you get $M^2$ congruent pieces.

![Scale by 3](image)

$3^2 \leftarrow$ congruent pieces

![Scale by 2](image)

$2^3 \leftarrow$ congruent pieces

(3) If you rescale a cube by a factor of $M$ you get $M^3$ congruent pieces.

**Scaling Dimension:** If there is a uniform scaling factor $M$ which transforms an object into $N$ (non-overlapping) pieces congruent to the original object, then the solution $d$ to the equation $N=M^d$, is the scaling dimension. Computationally, $d = \ln(M)/\ln(N)$. Note: $M$ and $N$ could be fractions rather than integers.
Why fractals are called fractals: their dimension is a real number which is usually not an integer!

**Exercises:** Find the scaling dimensions of the Cantor set, the Koch set $K$, and the Sierpinski triangle. Optional: find the scaling dimensions of the Sierpinski Pyramid and the Menger Sponge.