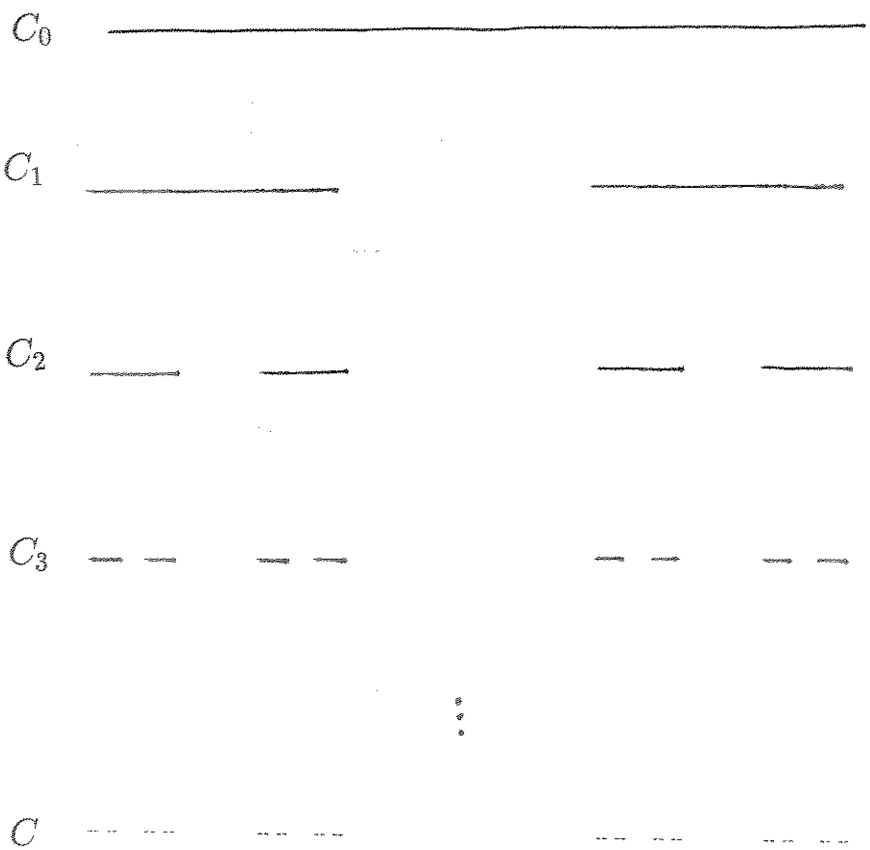
Part IV: Historical Fractals

Georg Ferdinand Ludwig Philipp Cantor
1845(Russia) – 1918(Germany)

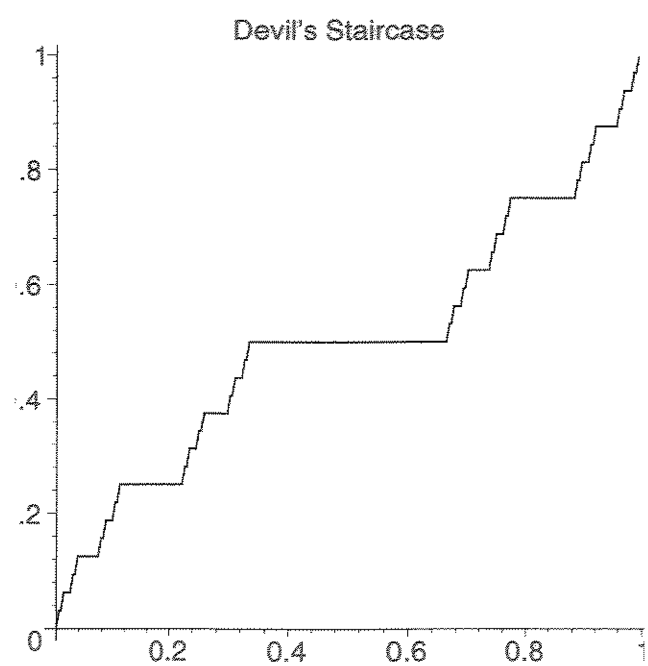
Best known for the Cantor set, which is constructed by starting with the unit interval, deleting the (open) middle 1/3 of this interval, and then successively deleting open middle thirds from the remaining set. The points which are never deleted in this process comprise the Cantor Set. A related function is the "Devil's Staircase" which is continuous and has derivative equal to zero on 100% (but not all!) of it's domain – yet is not constant. Cantor, who loved set theory and the logical underpinnings of mathematics, suffered from depression and mental illnesses in his later years.

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Cantor.html



Cantor set

How long is the Cantor Set?



Niels Fabian Helge von Koch
1870-1924 (Sweden)

Best known for the Koch snowflake, made as follows: begin with the unit interval. Delete the middle open third and replace with two intervals of length 1/3, forming two sides of an equilateral triangle. Iterate this process, at each stage replacing each edge with 4 sub-edges having 1/3 the original edge length!

Take the limit of this process (?!), and glue three of the limit pieces together in an equilateral fashion to create the Koch snowflake.

K_0

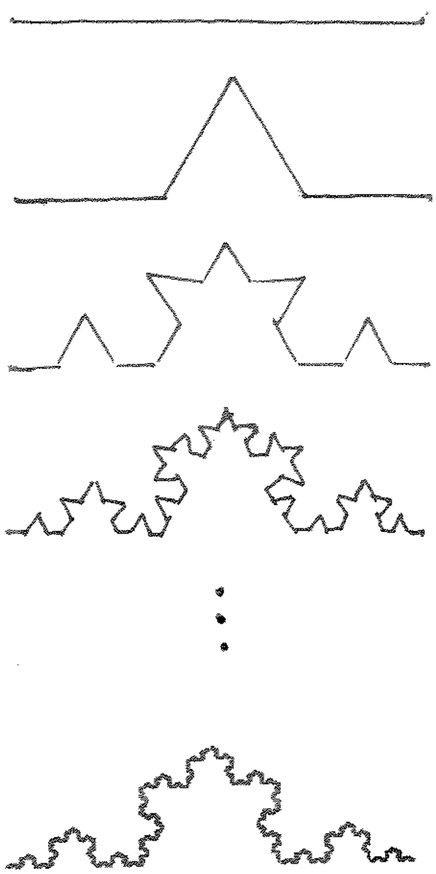
K_1

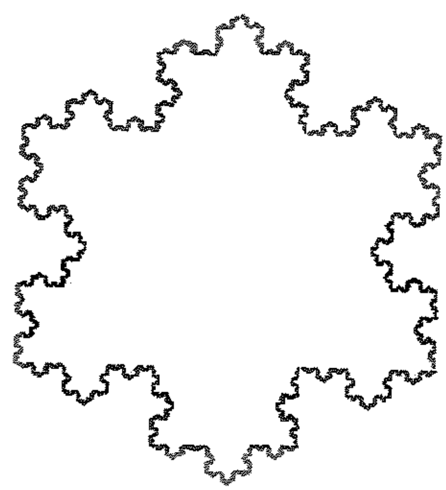
K_2

K_3

⋮

K





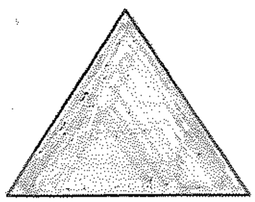
Koch Snowflake

How long is K?

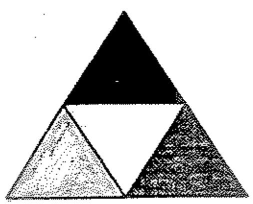
Waclaw Sierpinski
1870-1924 (Poland)

You construct the Sierpinski Triangle as follows:
Begin with the a closed equilateral triangular region,
Say with triangle edge lengths equal to 1.
At stage 1 subdivide this triangular region into four equilateral subregions, with new edges having half the previous side lengths. Remove the open central sub-triangular region (keeping the edges). Continue this process inductively. The Sierpinsiki Triangle is the collection of points from the original closed region which never get deleted.

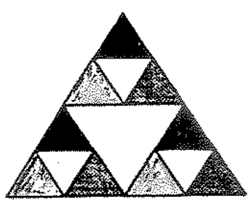
S_0



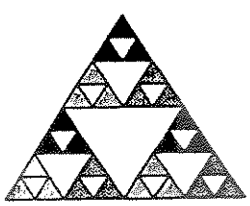
S_1



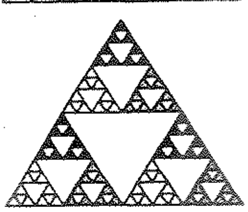
S_2



S_3

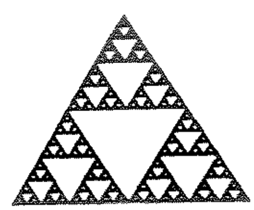


S_4



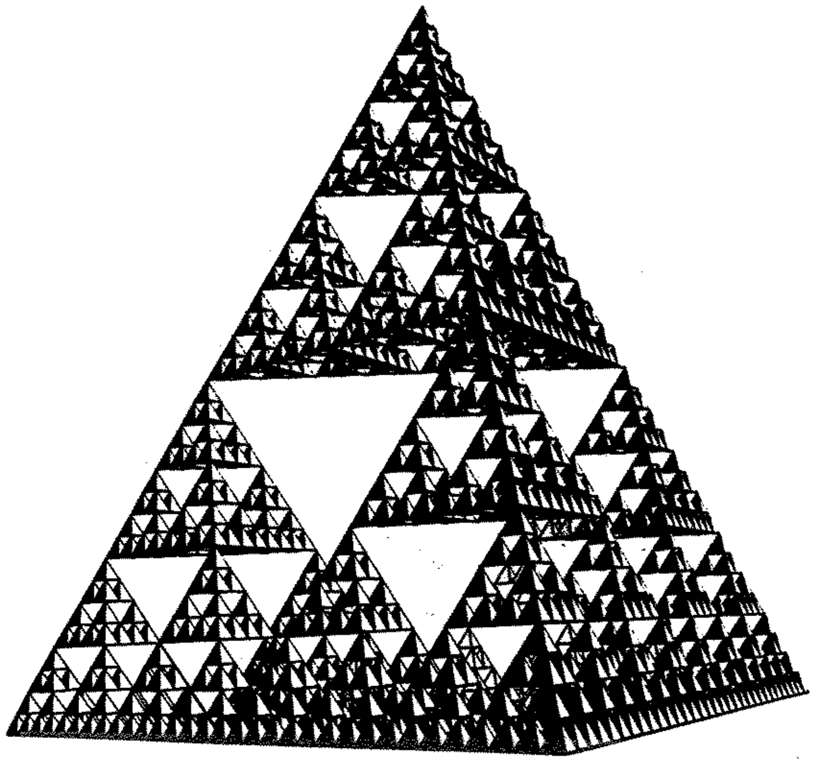
⋮

S



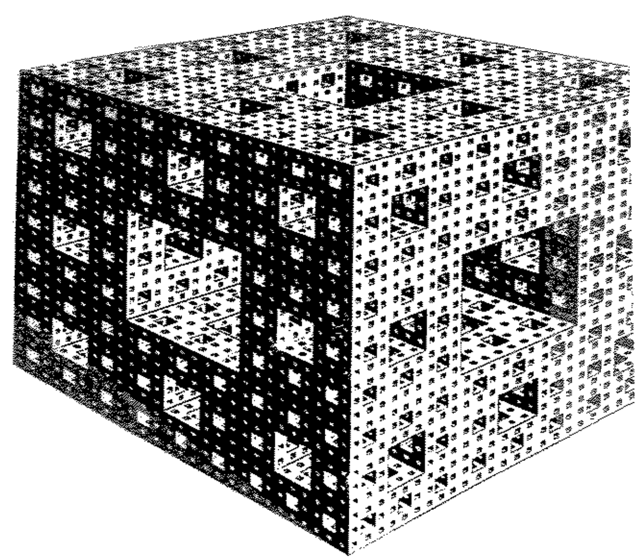
Sierpinski's Triangle
What is its area?

You can also make fractals in 3 dimensional space….



Sierpinski Pyramid

Karl Menger
1902 (Austria) -1985 (Chicago USA)



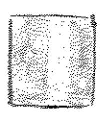
Menger Sponge
(How much volume?)

Scaling (Self-Similarity) Dimension

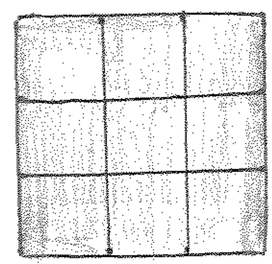
The concept scaling dimension builds on the ideas we used to understand how area, length, and volume of objects changed under uniform scaling of the background space. There is a much more general notion of dimension, called "Hausdorff dimension", which is also much harder to explain.

Examples:

(1) If you scale a line segment by an integer factor of M, you get M^1 pieces which are congruent to the original line segment.
(2) If you rescale a square by an integer factor of M, you get M^2 congruent pieces.

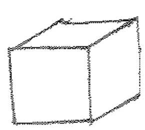


scale by 3 →

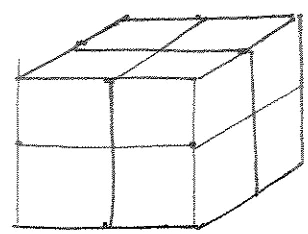


3^2 congruent pieces

(3) If you rescale a cube by a factor of M you get M^3 congruent pieces.



scale by 2 →



2^3 congruent pieces

<u>Scaling Dimension</u>: If there is a uniform scaling factor M which transforms an object into N (non-overlapping) pieces congruent to the original object, then the solution d to the equation $N=M^d$, is the scaling dimension. Computationally, $d = \ln(M)/\ln(N)$. Note: M and N could be fractions rather than integers.

Why fractals are called fractals: their dimension is a real number which is usually not an integer!

<u>Exercises:</u> Find the scaling dimensions of the Cantor set, the Koch set K, and the Sierpinski triangle. Optional: find the scaling dimensions of the Sierpinski Pyramid and the Menger Sponge.