Part II: Harder Transformations

Actually, they are not harder, they are just more general.

$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ax+cy+e \\ bx+dy+f \end{bmatrix}$$

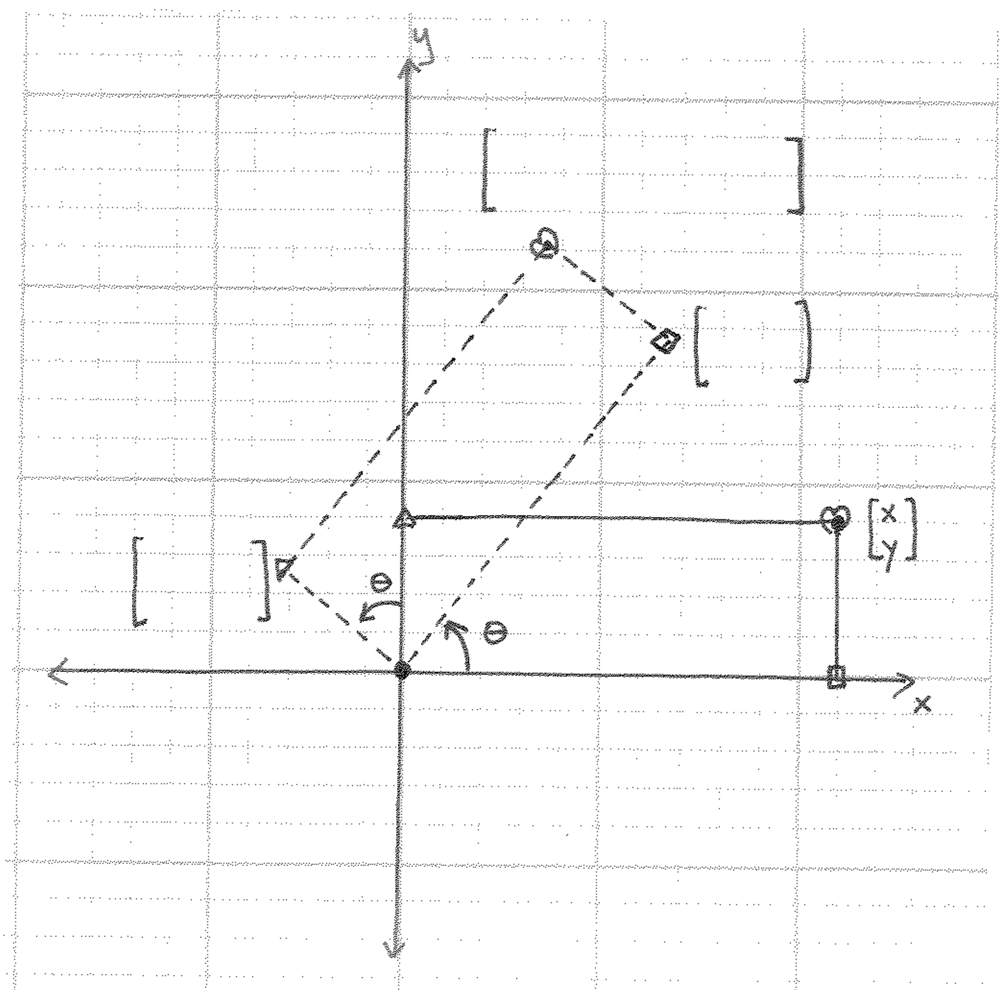
Transformations of this sort are called **affine**. These also include the easy transformations we talked of earlier.

1. Translation: $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix}$
2. Scaling and Translation: $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 5y \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix}$
3. Reflection: $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

etc.

There are also rotations. For example we could rotate by θ radians counterclockwise with the following formula:

$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

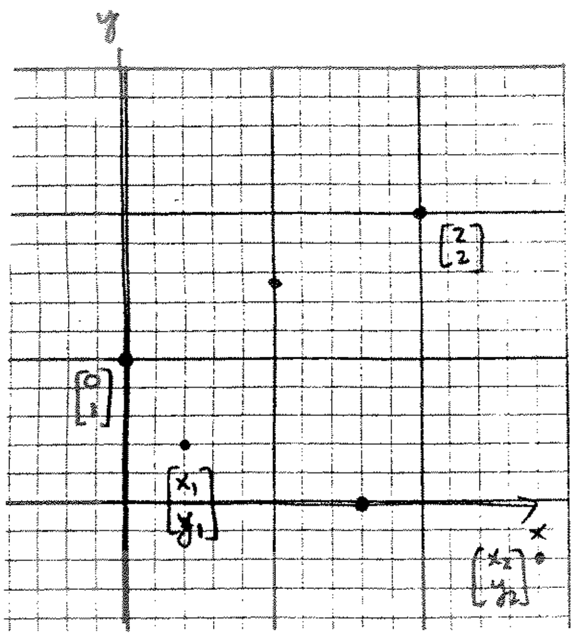


Part III: Geometry of affine transformations

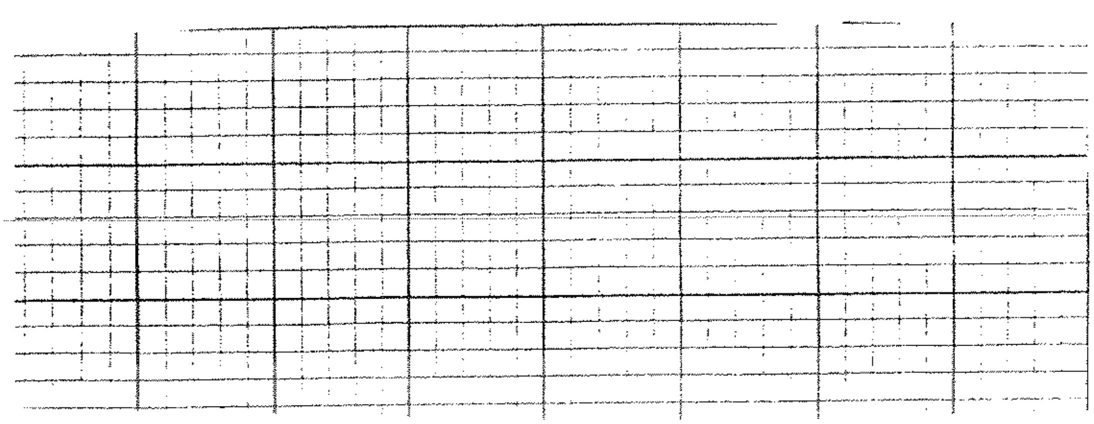
$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

Theorem 1. Let A be an affine map. Then:

1. For any two points P, Q and their midpoint M, the midpoint of $A(P)$ and $A(Q)$ is $A(M)$.



2. Affine maps transform lines into lines.

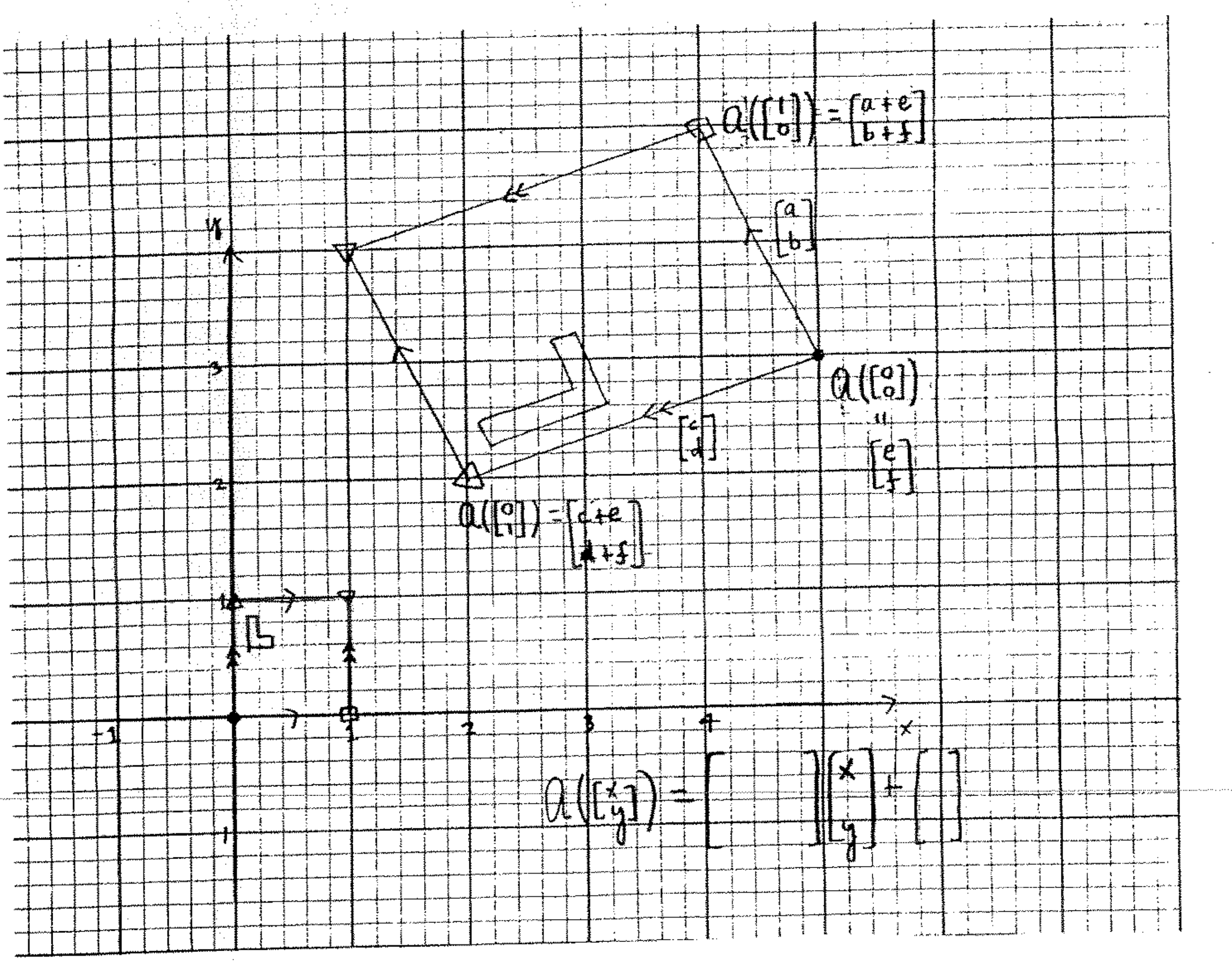


3. Affine maps transform rectangles into parallelograms, so they also transform rectangular grids into parallelogram grids.

Affine transformation template:

$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ax + cy + e \\ bx + dy + f \end{bmatrix}$$

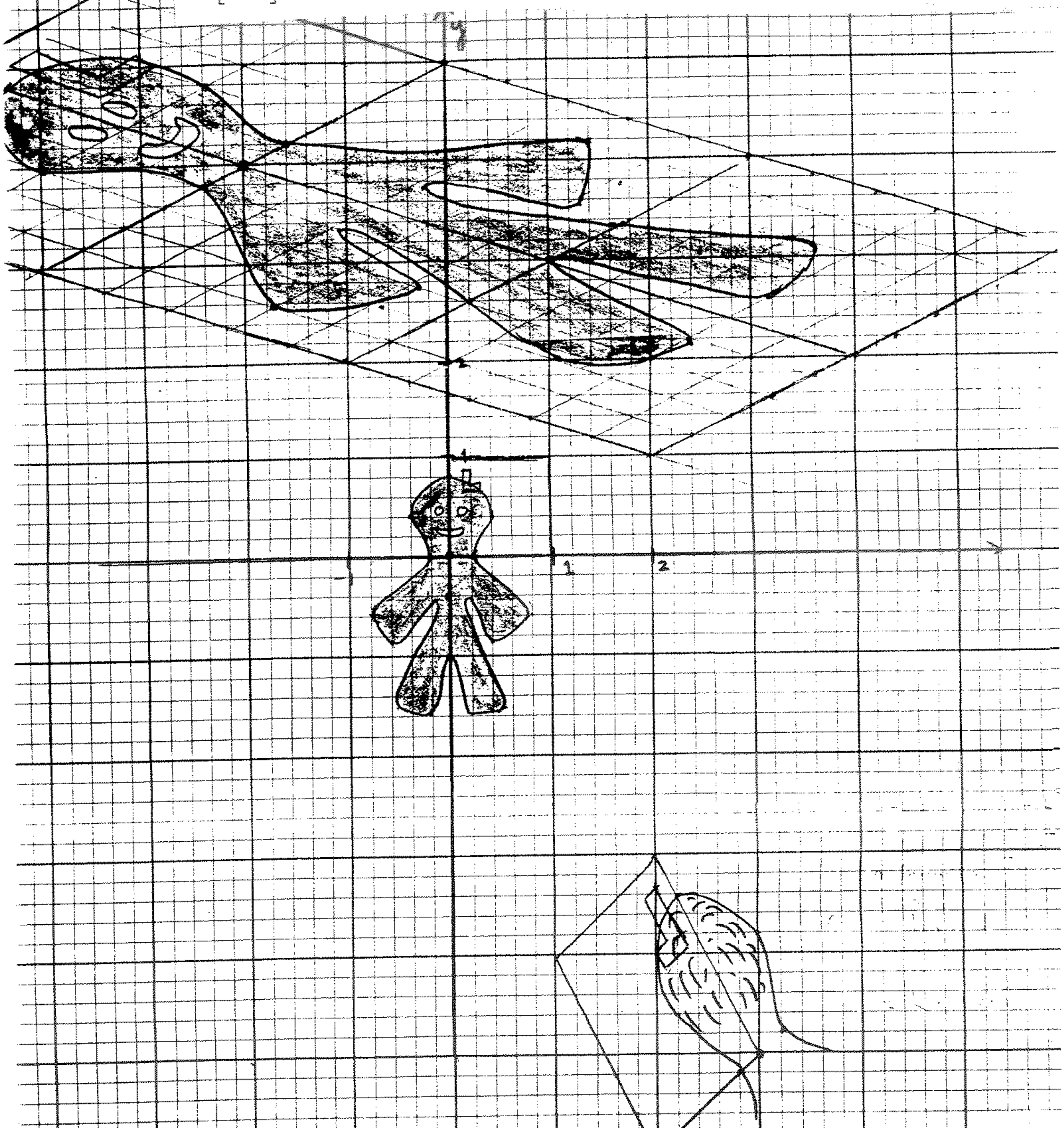
In this formula, $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ distorts Bob, and $\begin{bmatrix} e \\ f \end{bmatrix}$ translates him.



Bob and L-box transform themsleves:

Exercise 5. Find formulas for the two affine maps which are shown.

Exercise 6. Show where the L-box is transformed to by $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -3 \end{bmatrix}$. If you want, also draw in a piece of transformed Bob.



Exercise 7. How do general affine transformations affect area? How are the area of "Bob" and the transformed "A(Bob)" related? Hints: Since Bob is filled up with squares, and since the area of any square is multiplied by the same area "expansion" (or contraction) factor under an affine transformation, it suffices to find the area of the transformed unit square. Also, translations don't change area, so you may assume $\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so

$$A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Find the area of the parallelgram below (in terms of the letters a, b, c, d) to deduce your answer. Is your answer consistent with the earlier exercises in which you computed area expansion factors?