Two-Dimensional Poincaré-Hopf

Definition For a vector field V on a manifold M, let $Sing(V)$ be the set of points where $V(p) = 0$ and define for $p \in Sing(V)$: $I(V,p) = \dfrac{1}{2\pi}\displaystyle\int_{\partial D} \tilde{\beta}'(t)dt$ where D is a disc containing p such that p is the only singularity of V on D and $\tilde{\beta}(t) =$ angle between V and any nonzero vector field defined in a neighborhood of D.

Poincaré-Hopf Theorem: Let M be a closed orientable manifold of dimension 2 and V a vector field on M with isolated singularities. Then $\displaystyle\sum_{p \in Sing(V)} I(V,p) = \chi(M)$.

First we need something that will enable us to compute the Euler characteristic $\chi(M)$. For a triangle $\Delta \subset M$ we define $\beta(t) =$ angle between γ' and e_1 where γ parameterizes $\partial\Delta$ and $\{e_1, e_2\}$ constitute a local orthonormal frame field in a neighborhood of Δ. Then as a consequence of the turning tangents lemma: $\displaystyle\int_\Delta \beta'(t)dt =$ sum of the angles for $\Delta - \pi$.

Thus if we sum over a triangulation of M, M possibly with boundary:

Lemma: $\displaystyle\sum_{\Delta \subset M} \frac{1}{2\pi}\int_\Delta \beta'(t)dt = \chi(M)$.

Proof The sum on the left is $V_i + \frac{1}{2}V_\partial - \frac{1}{2}F$ where V_i is the number of interior vertices, V_∂ is the number of boundary vertices and F is the number of faces. As $\chi(S^1) = 0$, $V_\partial = E_\partial$ and $3F = 2E_i + E_\partial$ from considering the ∂-map so we get $V_i + \frac{1}{2}V_\partial - \frac{1}{2}F = V - \frac{1}{2}V_\partial - \frac{1}{2}F = V - \frac{1}{2}E_\partial - E_i - \frac{1}{2}E_\partial + F = V - E + F = \chi(M)$.

If D_p is the triangle containing $p \in Sing(V)$:

$$\frac{1}{2\pi}\int_{\partial D_p} \tilde{\beta}'(t) - \beta'(t)dt + 1 = I(V,p)$$

by the turning tangents lemma. Thus:

$$\frac{1}{2\pi}\sum_{p \in Sing(V)}\int_{\partial D_p} \tilde{\beta}'(t) - \beta'(t)dt + n = \sum_{p \in Sing(V)} I(V,p)$$

where n is the number of singularities. Let M' be the manifold, possibly with boundary, obtained by deleting $\mathrm{int}(D_p)$, where D_p is a disc which contains only the singularity p, for each $p \in Sing(V)$. Thus $\chi(M) = \chi(M') + n$. Since $\displaystyle\bigcup_{p \in Sing(V)} \partial D_p = \partial M'$ with the opposite orientation:

$$\frac{1}{2\pi}\sum_{p\in Sing(V)}\int_{\partial D_p}\tilde{\beta}'(t)-\beta'(t)dt+n=\frac{1}{2\pi}\int_{\partial M'}\beta'(t)-\tilde{\beta}'(t)dt+n$$

We can then break this up into a sum of integrals using a triangulation on M':

$$\frac{1}{2\pi}\int_{\partial M'}\beta'(t)-\tilde{\beta}'(t)dt+n=\frac{1}{2\pi}\sum_{\Delta\subset M'}\int_{\partial M'}\beta'(t)-\tilde{\beta}'(t)dt+n$$

However, when computing $\tilde{\beta}$ on M', we may use the vector field V for our background vector field as it in nonzero in a neighborhood of the triangles, thus $\tilde{\beta}=0$ on M' and:

$$\frac{1}{2\pi}\sum_{\Delta\subset M'}\int_{\partial M'}\beta'(t)-\tilde{\beta}'(t)dt+n=\frac{1}{2\pi}\sum_{\Delta\subset M'}\int_{\partial M'}\beta'(t)dt+n=\chi(M')+n=\chi(M)$$

completing the proof.