Math 5440
Wednesday Sept 8

- Last Friday we discussed multivariable FTC
  still need to show how it implies Green's, div (& Stokes') theorems.
  if interest, we can also do an example of Green's, as a way to recall <u>line ~~ta~~ integrals</u>.

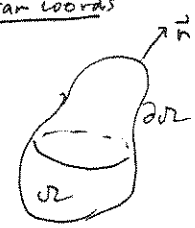
- Set up Navier Stokes derivation problem on your hw.
  this will let us review one interpretation of <u>flux integrals</u>

<u>flux integrals for fluid motion</u>

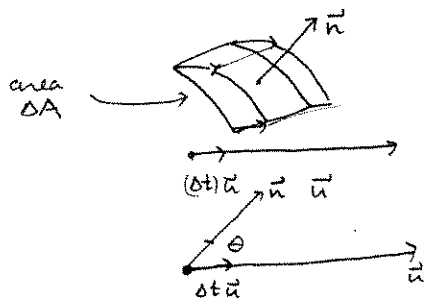
Let fluid motion have velocity field $\vec{u}(\vec{x},t)$, density $\rho(\vec{x},t)$ mass/vol.

fixed (non-moving) background coords,
called <u>Eulerian coords</u>

Let Ω be a test domain (fixed),
with boundary surface $\partial\Omega$.
<u>How fast is mass flowing thru bdry?</u>



look at a small piece of bdry:



in time Δt an approximate parallelepiped slab of thickness $\Delta t\, \vec{u}\cdot\vec{n}$, so volume $(\Delta A)\vec{u}\cdot\vec{n}\,\Delta t$ & mass $\rho(\Delta A)\vec{u}\cdot\vec{n}(\Delta t)$ flows through bdry piece.

keeping track of signs, the mass flow <u>into</u> Ω in time Δt

$$\approx \Delta t\left(-\iint_{\partial\Omega} \rho\, \vec{u}\cdot\vec{n}\, dA\right)$$

$$\boxed{\text{the flux of mass into } \Omega \text{ is } \quad -\iint_{\partial\Omega} \rho\, \vec{u}\cdot\vec{n}\, dA \quad \text{mass/time.}}$$

HW 3a)
Conservation of mass (assuming none is being created)

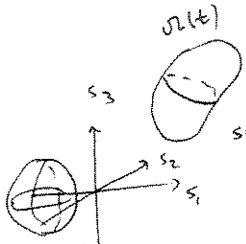
$$\underbrace{\frac{d}{dt}\iiint_{\Omega} \rho \, dV}_{\text{rate of change of total mass}} = \underbrace{-\iint_{\partial\Omega} \rho \vec{u}\cdot\vec{n} \, dA}_{\text{how fast mass is entering } \Omega \text{ thru } \partial\Omega.}$$

leads to continuity equation $\boxed{\nabla\cdot(\rho\vec{u}) + \rho_t = 0}$

HW 3b) As in week 1 HW, it's easier to use a coord system (s_1, s_2, s_3) of particles moving with the fluid, i.e.

$$\frac{\partial}{\partial t}\vec{x}(s,t) = \vec{u}(\vec{x}(s,t),t)$$

This is called the Lagrangian description



snapshot of moving distorted ball at time t

ball if system at rest

$\vec{u}(\vec{x}(s,t),t)$
$P(\vec{x}(s,t),t)$
$\rho(\vec{x}(s,t),t)$
$\vec{F}(\vec{x}(s,t),t)$

measure these quantities for a particle moving with fluid which would be at location $\vec{s}$ if fluid were at rest.

Newton's 2nd law:

change in total momentum = net forces:

- $$\frac{d}{dt}\iiint_{s\text{-ball}} \rho_0 \vec{u}(\vec{x}(s,t),t) \underbrace{dV_s}_{ds_1ds_2ds_3} = -\iint_{\partial(\text{moving } \Omega(t))} p\vec{n} \, dA + \iiint_{s\text{-ball}} \vec{F}\rho_0 dV_s$$

  $\vec{F}(\vec{x}(s,t),t)$

$\frac{D}{Dt}($~~[illegible]~~ $g(x,t))$
$:= \nabla g\cdot\vec{u} + g_t$

grad wrt x, y, z only

is called the material derivative of g, and measures how fast the quantity g is changing for fixed particle.

key example:
$\frac{D}{Dt}\vec{u} = (\nabla_x \vec{u})\cdot\vec{u} + u_t$
is particle acceleration!

- relationship of ρ to ρ_0

  $\rho(\vec{x}(s,t),t)$

mass in equil = mass in moving fluid.

$$\rho_0 dV_s = \rho \, J \, dV_s$$

J: Jacobian det $\frac{\partial(x,y,z)}{\partial(s_1,s_2,s_3)}$ volume expansion factor

so $\boxed{\rho J = \rho_0}$

Leads to Navier Stokes: $\boxed{\underbrace{\rho\vec{u}_t + \rho\vec{u}\cdot\nabla\vec{u}}_{\rho\frac{D}{Dt}\vec{u}} = -\nabla p + \rho\vec{F}}$

(mass)(acceleration) = net forces!

Linearization to multivariable wave equation

$\rho = \rho_0 + O(\varepsilon)$, $\rho_t, \nabla_x \rho = O(\varepsilon)$
$\vec{u}, \vec{u}_t, \nabla_x \vec{u} = O(\varepsilon)$
$\vec{F} = O(\varepsilon)$
$p = p_0 + O(\varepsilon)$, $\nabla_x p, \nabla_t p = O(\varepsilon)$

$$\nabla\cdot(\rho\vec{u}) + \rho_t = 0 \Rightarrow \underbrace{\nabla\rho\cdot\vec{u}}_{O(\varepsilon^2)} + \underbrace{\rho}_{\rho_0+O(\varepsilon)} \operatorname{div}\vec{u} + \rho_t = 0 \xrightarrow{\text{linearize}} \boxed{\rho_0 \operatorname{div}\vec{u} + \rho_t = 0} \quad (1)$$

$$\underbrace{\rho}_{\rho_0+O(\varepsilon)}\vec{u}_t + \rho\,\underbrace{\vec{u}\cdot\nabla\vec{u}}_{O(\varepsilon^2)} + \nabla p = \underbrace{\rho}_{\rho_0+O(\varepsilon)}\vec{F} \xrightarrow{\text{linearize}} \rho_0\vec{u}_t + \nabla p = \rho_0\vec{F}$$

if $p = p(\rho)$ then $\nabla p = p'(\rho)\nabla\rho \xrightarrow{\text{linearize}} p'(\rho_0)\nabla\rho$

$$\boxed{\rho_0\vec{u}_t + p'(\rho_0)\nabla\rho = \rho_0\vec{F}} \quad (2)$$

As before, take cross derivs & add

$$\partial_t (1) - \operatorname{div} (2) \Rightarrow \boxed{\rho_{tt} - c^2\Delta\rho = -\rho_0 \operatorname{div}\vec{F}}$$ multivariable wave eqtn.

$c^2 = p'(\rho_0)$
$\Delta\rho = \operatorname{div}(\nabla\rho) = \rho_{xx} + \rho_{yy} + \rho_{zz}$ "Laplacian"

(multivariable wave eqtn for $\vec{u}$ needs more assumptions & work).

for $\rho_{tt} - c^2\rho_{xx} = 0$ one space variable, soltns are superposition $p(x+ct) + q(x-ct)$

for $\rho_{tt} - c^2\Delta u = 0$ multi space variable, soltns turn out to be integral superpositions of plane waves traveling in all possible directions !!