Integration

Basic integration

$\int_a^b f(x)\,dx$

double or triple integrals over
rectangles or coordinate boxes

more complicated iterated integrals in $\mathbb{R}^2$, $\mathbb{R}^3$

domain $\leftrightarrow$ setting up iterated integrals

area, volume, mass, center of mass, moments of inertia
(memorize or write down formulas!)

Polar coords
$dA = r\,dr\,d\theta$

Cylindrical coords
$dV = r\,dr\,d\theta\,dz$

Spherical coords
$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

More complicated integrals

ALL based on small scale (tangent) approximation and parameterization

$$\int_C f(\vec{x})\,ds := \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

$\vec{x} = \vec{r}(t)$ parametric curve

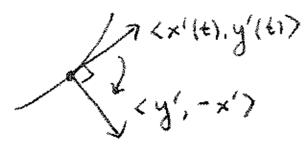
$d\vec{r} = \vec{r}\,'(t)\,dt$ tangent displacement

$ds = \|d\vec{r}\| = \|\vec{r}\,'(t)\|\,dt$ tangent distance (element of arclength)

$$\int_C \vec{F}\cdot d\vec{r} := \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

also written as $\int_C \vec{F}\cdot\vec{T}\,ds$, $\int_C M\,dx + N\,dy + P\,dz$

in $\mathbb{R}^2$, $\int_C \vec{F}\cdot\vec{n}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\langle y', -x'\rangle\,dt$



$\mathbb{R}^2$
$d\vec{r} = \vec{T}\,ds = \langle x', y'\rangle\,dt$
$\vec{n}\,ds = \langle y', -x'\rangle\,dt$

$$\underset{\substack{\uparrow\\ \text{graph}}}{\iint_S} f\,dS = \underset{\substack{\uparrow\\ \text{x-y region}}}{\iint_R} f(x,y,g(x,y))\underbrace{\sqrt{1+g_x^2+g_y^2}\,dA}_{dS \text{ for a graph } z=g(x,y)}$$

$$\underset{\text{graph}}{\iint_S} \vec{F}\cdot\vec{n}\,dS = \underset{\text{x-y region}}{\iint_R} \vec{F}(x,y,g(x,y))\cdot\langle -g_x, -g_y, 1\rangle\,dx\,dy$$

$dS = \sqrt{1+g_x^2+g_y^2}\,dA$, $\vec{n} = \dfrac{\langle -g_x, -g_y, 1\rangle}{\sqrt{1+g_x^2+g_y^2}}$

$\vec{n}\,dS = \langle -g_x, -g_y, 1\rangle\,dA$
(cancellation!)

or for general parametric surfaces

$$\underset{\substack{\uparrow\\ \text{parameterized surface}}}{\iint_S} f\,dS = \underset{\substack{\uparrow\\ \text{u-v parameterization domain}}}{\iint_R} f(\vec{r}(u,v))\,\|\vec{r}_u\times\vec{r}_v\|\,du\,dv$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$$

$dS = \|\vec{r}_u\times\vec{r}_v\|\,du\,dv$

$\vec{n} = \dfrac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|}$

$\vec{n}\,dS = (\vec{r}_u\times\vec{r}_v)\,du\,dv$

- The integral of a constant over a curve, 2-dim'l object, or 3-dim'l object yields that constant times the length, surface area, or volume, respectively!!

Fundamental Theorem(s) of Calculus

$\mathbb{R}^1$: $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$

$\mathbb{R}^2$: $\displaystyle\iint_D D_{\vec{u}} f\, dA = \int_{\partial D} f\, \vec{u}\cdot\vec{n}\, ds$

$\mathbb{R}^3$: $\displaystyle\iiint_S D_{\vec{u}} f\, dV = \iint_{\partial D} f\, \vec{u}\cdot\vec{n}\, dS$

$\mathbb{R}^1$ apps:

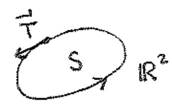
If $\vec{F} = \nabla f$ then $\displaystyle\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt = \int_a^b \underbrace{\nabla f(\vec{r}(t))\cdot\vec{r}'(t)}_{\frac{d}{dt} f(\vec{r}(t))}\,dt = f(\vec{r}(b)) - f(\vec{r}(a))$

$= f(B) - f(A)$

When is $\vec{F}$ a gradient field ∇f?

(curl test)

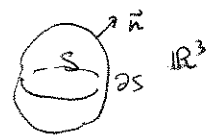
How to find f in this case.



$\mathbb{R}^2$ apps: Green's Thm: $\displaystyle\oint_{\mathcal{C}} \vec{F}\cdot\underbrace{\vec{T}\,ds}_{d\vec{r}} = \iint_S N_x - M_y\, dA$

(& $\mathbb{R}^3$) Stoke's $\displaystyle\oint_{\partial S} \vec{F}\cdot\vec{T}\,ds = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dS$

div Thm: $\displaystyle\int_{\partial S} \vec{F}\cdot\vec{n}\,ds = \iint_S \vec{\nabla}\cdot\vec{F}\, dA$



$\mathbb{R}^3$ app div Thm: $\displaystyle\iint_{\partial S} \vec{F}\cdot\vec{n}\, dS = \iiint_S \vec{\nabla}\cdot\vec{F}\, dV$

be able to check either side of these FTC's,
by computing the integrals.

The Basic (any variable) FTC.

(and how it implies div thm in any $\mathbb{R}^n$)

$$\boxed{n=1 \quad \int_a^b F'(x)\,dx = F(b)-F(a)}$$

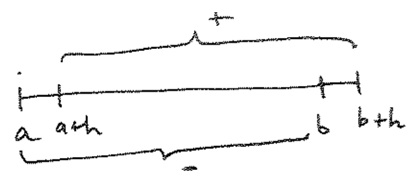
usually we write $f(x)$ for $F'(x)$

nifty way to see this: let $h>0$ small

$$\int_a^b F'(x)\,dx \approx \int_a^b \frac{F(x+h)-F(x)}{h}\,dx$$

$$= \frac{1}{h}\left[\int_a^b F(x+h)\,dx - \int_a^b F(x)\,dx\right]$$

$u = x+h$
$du = dx$



$$= \frac{1}{h}\left[\int_{a+h}^{b+h} F(u)\,du - \int_a^b F(x)\,dx\right]$$

notice $\int_{a+h}^{b} F(x)\,dx$ cancels

$$= \frac{1}{h}\left[\int_b^{b+h} F(x)\,dx - \int_a^{a+h} F(x)\,dx\right]$$

$$\approx F(b)-F(a). \quad \blacksquare$$

It's possible to turn this into a proof, by letting $h \to 0$.

$n=2$ let $\vec{u}$ be any unit vector.
Let S be a region bounded by a curve $\mathcal{C}$, with unit normal (exterior) $\vec{n}$
Then if f is differentiable on S,

$$\boxed{n=2 \quad \iint_S D_{\vec{u}} f\,dA = \int_{\mathcal{C}} f\,\vec{u}\cdot\vec{n}\,ds}$$

$n=3$: let $\vec{u}$ be any unit vector.
Let $S \subset \mathbb{R}^3$ be a region bounded by a surface ∂S, with unit exterior normal $\vec{n}$
Then for differentiable f,

$$\boxed{n=3 \quad \iiint_S D_{\vec{u}} f\,dV = \iint_{\partial S} f\,\vec{u}\cdot\vec{n}\,dS}$$

In fact, the analogous theorem is true in any space dimension, and the case $n=1$ is included, since $F'(x) = D_u f$ for $u = +1$.

Math 2210 Korevaar
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The $n=2$ picture indicates why this FTC is true in all space dimensions

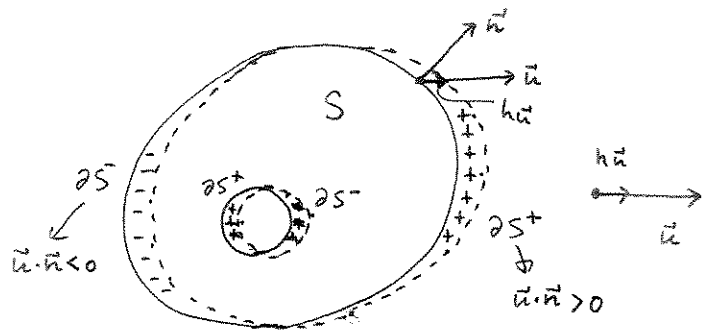
$$\boxed{\iint_S D_{\vec{u}} f \, dA} \approx \iint_S \frac{f(\vec{x}+h\vec{u})-f(\vec{x})}{h} \, dA$$

$$= \frac{1}{h}\left[\iint_S f(\vec{x}+h\vec{u})\, dA - \iint_S f(\vec{x})\, dA\right.$$

translation
COV
$\vec{z} = \vec{x} + h\vec{u}$
leaves dA unchanged.

$$* \qquad = \frac{1}{h}\Bigg[\iint_{\underbrace{S+h\vec{u}}_{\text{S translated by } h\vec{u}}} f(\vec{z})\, dA - \iint_S f(\vec{x})\, dA\Bigg]$$

integral's away from ∂S cancel out,
leaving S_h^+ with + contribution,
S_h^- with − contribution



$$* = \frac{1}{h}\iint_{S_h^+} f(\vec{x})\underbrace{dA}_{\approx\, h\vec{u}\cdot\vec{n}\, ds} - \frac{1}{h}\iint_{S_h^-} f(\vec{x})\underbrace{dA}_{\approx\, (-h\vec{u}\cdot\vec{n})\, ds}$$

$$\approx \frac{1}{\cancel{h}}\int_{\partial S^+} f\, \overbrace{\cancel{h}\vec{u}\cdot\vec{n}\, ds} - \frac{1}{\cancel{h}}\int_{\partial S^-} f\, \overbrace{(-h\vec{u}\cdot\vec{n})\, ds}$$

$$\boxed{= \int_{\partial S} f\, \vec{u}\cdot\vec{n}\, ds} \quad \blacksquare$$

One can take $\lim_{h\to 0}$ and create an actual proof, although it's technically difficult so textbooks use iterated integrals and decomposition ideas like we did for Green's Theorem

for $n=3$, do you see how the argument would go?

$n=2 \quad n=3$

§14.4, 14.6

FTC $\Longrightarrow$ divergence Theorem

Monday

$n=2$ divergence $\Longrightarrow$ Green's. $\longrightarrow$ Stoke's

14.4   14.7

$\underline{n=2}$: let $\vec{F} = \langle M, N \rangle$

FTC, $\vec{u} = \langle 1, 0 \rangle$: $$\iint_S \frac{\partial M}{\partial x}\, dA = \int_{\mathcal{C}} M \langle 1,0\rangle \cdot \vec{n}\, ds$$

$\vec{u} = \langle 0, 1 \rangle$: $$\iint_S \frac{\partial N}{\partial y}\, dA = \int_{\mathcal{C}} N \langle 0,1\rangle \cdot \vec{n}\, ds$$

$\underline{\text{add}}$: $$\boxed{\iint_S \vec{\nabla}\cdot\vec{F}\, dA = \int_{\mathcal{C}} \langle M,N\rangle \cdot \vec{n}\, ds} \quad ∎$$

$n=3$: let $\vec{F} = \langle M, N, P \rangle$

FTC, $\vec{u} = \hat{\imath}$: $$\iiint_S M_x\, dV = \iint_{\partial S} M \langle 1,0,0\rangle \cdot \vec{n}\, dS = \iint_{\partial S} \langle M,0,0\rangle \cdot \vec{n}\, dS$$

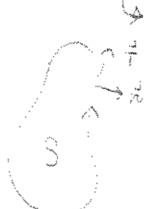
$\vec{u} = \hat{\jmath}$: $$\iiint_S N_y\, dV = \iint_{\partial S} N \langle 0,1,0\rangle \cdot \vec{n}\, dS$$

$\vec{u} = \hat{k}$: $$\iiint_S P_z\, dV = \iint_{\partial S} P \langle 0,0,1\rangle \cdot \vec{n}\, dS$$

$\underline{\text{add}}$: $$\boxed{\iiint_S \vec{\nabla}\cdot\vec{F}\, dV = \iint_{\partial S} (\vec{F}\cdot\vec{n})\, dS}$$

<u>etc.</u>

$n=2$ Green's Theorem from $n=2$ div Thm:



Says $\vec{F} = \langle M, N \rangle$

$$\iint_S N_x - M_y\, dA = \int_{\partial S} M\,dx + N\,dy$$

proof: $$\iint_S N_x - M_y\, dA = \iint_S \operatorname{div}\langle N, -M\rangle\, dA$$

$$\overset{n=2\,\text{div}}{=} \int_{\partial S} \langle N, -M \rangle \cdot \vec{n}\, ds$$

$$\overset{\text{geom}}{=} \oint_{\partial S} \langle M, N\rangle \cdot \underbrace{\vec{T}\,ds}_{\langle dx,\, dy\rangle} = \oint_{\partial S} M\,dx + N\,dy \quad ∎$$