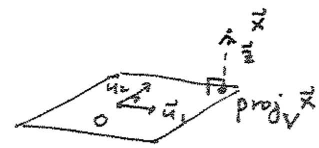
Math 5440
Friday 10/8 Chapter 4 cont'd

- Finish discussing heat equation on Wed. notes, in conjunction with today's.

Recall the following Theorems from 2270-2280

<u>Theorem 1</u> Let $\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k\}$ be an orthonormal basis for a k-dim'l subspace V of $\mathbb{R}^n$

Then
$$\text{proj}_V \vec{x} := \sum_{j=1}^{k} (\vec{u}_j \cdot \vec{x}) \vec{u}_j$$



has the property that $\vec{z} := \vec{x} - \text{proj}_V \vec{x}$ is $\perp$ to V.

Therefore $\text{proj}_V \vec{x}$ is the nearest point to $\vec{x}$ in V.

(If $k=n$, then $\vec{x} = \sum_{j=1}^{n} (\vec{u}_j \cdot \vec{x}) \vec{u}_j$.)

(we may wish to recall the proof)

<u>Def</u> A (real) inner product space is a vector space V (with real scalars) together with an <u>inner product</u> $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ s.t. $\forall f, g, h \in V$, $c \in \mathbb{R}$

a) $\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$
b) $\langle cf, h \rangle = c \langle f, h \rangle$
c) $\langle f, g \rangle = \langle g, f \rangle$

} symmetric and bilinear

d) $\langle f, f \rangle \geq 0$; $\langle f, f \rangle = 0$ iff $f = 0$.

"perpendicular"

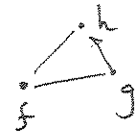
<u>Def</u> If $(V, \langle \cdot, \cdot \rangle)$ is a (real) inner product space then $f \perp g$ iff $\langle f, g \rangle = 0$

$\|f\| := \langle f, f \rangle^{1/2}$

"norm"

(for function space example, go to page 3)

Theorem 2 If ~~*~~ $(V, \langle \cdot, \cdot \rangle)$ is an inner product space then the distance function $d(f,g) := \|f-g\|$ satisfies



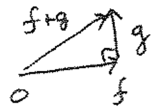
△ inequality

a) $d(f,h) \leq d(f,g) + d(g,h)$

b) $d(f,g) \geq 0$. $d(f,g) = 0$ iff $f = g$.

Furthermore, the norm function $\|\cdot\|$ satisfies the Pythagorean theorem

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2 \quad \text{iff } f \perp g$$



and we have the Cauchy-Schwartz inequality $|\langle f, g \rangle| \leq \|f\| \|g\|$ (pfs: see HW)

Theorem 3 Let $V \subset W$ be a finite dimensional subspace of an inner product space $(W, \langle \cdot, \cdot \rangle)$

Let $\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k\}$ be an orthonormal basis of V.

Then $\text{proj}_V f := \sum_{k=1}^{n} \langle f, u_k \rangle u_k$ has the property that

$$z := f - \text{proj}_V f \quad \text{is } \perp V.$$

Therefore, $\text{proj}_V f$ is the nearest point to f in V

Theorem 4 Let $V = \{ f : \mathbb{R} \to \mathbb{R}$ s.t. f is piecewise continuous and 2ℓ-periodic$\}$.
Let $\langle f, g \rangle := \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) g(x) dx$. This is an inner product on V.

Then the collection

$$\left\{ \frac{1}{\sqrt{2}},\ \cos\left(\frac{\pi}{\ell}x\right), \sin\left(\frac{\pi}{\ell}x\right), \cos\left(\frac{2\pi}{\ell}x\right), \sin\left(\frac{2\pi}{\ell}x\right), \dots \cos\left(\frac{n\pi}{\ell}x\right), \sin\left(\frac{n\pi}{\ell}x\right), \dots \right\}$$

is orthonormal

(there's a tedious proof and a slick proof).

uses addition angle formulas from trig

Let's do this one!

- the vectors in this list all have norm $= 1$

- they're mutually perpendicular: hint: use the fact that these are eigenfunctions of the 2nd derivative operator (Laplacian in 1 space dim.), and the formula

$$\int_{-\ell}^{\ell} f''(x) g(x) - f(x) g''(x) = f'g - fg' \Big]_{-\ell}^{\ell}$$

would this generalize to higher space dimensions?

<u>Corollary</u> Let $V_n = \text{span}\left\{ \frac{1}{\sqrt{2}}, \cos\frac{\pi}{\ell}x, \sin\frac{\pi}{\ell}x, \dots \cos\frac{n\pi}{\ell}x, \sin\frac{n\pi}{\ell}x \right\}$.

Let $f \in V$. Let $a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx = \langle f, 1\rangle$

$$a_k = \frac{1}{\ell}\int_{-\ell}^{\ell} \cos\left(\tfrac{k\pi}{\ell}x\right) dx = \langle f, \cos\left(\tfrac{k\pi}{\ell}x\right)\rangle \qquad k \geq 1$$

$$b_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\tfrac{k\pi}{\ell}x\right) dx = \langle f, \sin\left(\tfrac{k\pi}{\ell}x\right)\rangle$$

then
$$\text{proj}_{V_n} f = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + \sum_{k=1}^{n} b_k \sin kx$$

is the nearest point to f in V_n

Precisely, $\|f - \text{proj}_{V_n} f\|^2 \leq \|f - g\|^2 \quad \forall g \in V_n$, equality iff $g = \text{proj}_{V_n} f$.

$$\|f-g\|^2 = \frac{1}{\ell}\int_{-\ell}^{\ell} (f(x) - g(x))^2\,dx$$

<u>Def</u>. Let $f \in V$

the Fourier series for f is
$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\tfrac{k\pi}{\ell}x\right) + \sum_{k=1}^{\infty} b_k \sin\left(\tfrac{k\pi}{\ell}x\right)$$

convergence questions:

- Does $\text{proj}_{V_n} f \to f$ in the "mean square" integral d, $\forall f \in V$?
- for which x does $(\text{proj}_{V_n} f)(x) \to f(x)$? i.e. pointwise convergence.
- for which f does $\text{proj}_{V_n} f \to f$ uniformly?