Math 5440
Monday Oct. 4
Exam 1: Do 4 of the 5 problems below. Each problem is worth 25 points.
If you try all 5, indicate clearly which 4 you want graded

1) Consider the constant speed wave equation in $\mathbb{R}^2$,

$$u_{tt} - c^2 u_{xx} = 0$$

prove that any C^2 soln to this equation must be of the form

$$u(x,t) = p(x+ct) + q(x-ct)$$

where p, q are functions of one variable. Hint: factor the corresponding linear operator and change variables.

2) Consider the initial boundary value problem

$$\text{IBVP} \begin{cases} u_{tt} - 4u_{xx} = 1 & 0<x<2,\ t>0 \\ u(x,0) = 0 & \\ u_t(x,0) = \sin\frac{\pi}{2}x & 0<x<2 \\ u(0,t) = u(2,t) = 0 & t \geq 0 \end{cases}$$

Find $u(1,1)$.

3) Let Ω be a bounded domain in $\mathbb{R}^n$, with piecewise smooth $\partial\Omega$.
Use an integral argument to prove that solutions to the Dirichlet problem, as well as those to the Neumann problem, for Laplace's equation, are unique:

$$\text{DP} \begin{cases} \Delta u = F & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases} \qquad \text{NP} \begin{cases} \Delta u = F & \text{in } \Omega \\ Du \cdot \hat{n} = g & \text{on } \partial\Omega \end{cases}$$

$\hat{n}$ = unit exterior normal.

in both cases assume
$u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

4) The maximum principle for the heat equation states that for an open, bounded domain $\Omega \subset \mathbb{R}^n$ and a function $w \in C^2(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$, if

$$w_t - k\Delta w \leq 0 \quad \text{in } \Omega \times (0,T]$$

then the maximum of w occurs on the parabolic boundary $(\Omega \times \{0\}) \cup (\partial\Omega \times [0,T])$.

4a) Prove the maximum principle, in the "easy case", that $w_t - k\Delta w < 0$. (15 points)

4b) Use the maximum principle to prove that solutions to the Dirichlet problem

$$\begin{cases} u_t - k\Delta u = 0 & \text{in } \Omega \times (0,\infty) \\ u(x,t) = \varphi(x,t) & x \in \partial\Omega,\ t \geq 0 \\ u(x,0) = f(x) & x \in \bar{\Omega} \end{cases}$$

$u \in C^2(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$
φ continuous
f continuous.

are unique. (10 points)

5) We've studied the system of PDE's for the Navier Stokes' eqtns. It is

$$\begin{cases} \operatorname{div}(\rho\vec{u}) + \rho_t = 0 & \text{continuity eqtn} \\ \rho\vec{u}_t + \rho\,\vec{u}\cdot\nabla\vec{u} = -\nabla p + \rho\vec{F} & \text{Navier Stokes} \end{cases}$$

$\rho(\vec{x},t)$ = density
$p(\vec{x},t)$ = pressure
$\vec{F}(\vec{x},t)$ = body force/mass
$\vec{u}(\vec{x},t)$ = velocity field of particle motion.

5a) The continuity equation is the easier of the two to derive... derive it. (15 points)

5b) We linearized these equations, under the further assumption that pressure was a function of density. We arrived at

$$\begin{cases} \rho_0 \operatorname{div}\vec{u} + \rho_t = 0 \\ \rho_0 \vec{u}_t + p'(\rho_0)\nabla\rho = \rho_0\vec{F} \end{cases}$$

Use this system to show that density $\rho(\vec{x},t)$ in this model satisfies the 3 space dimension wave equation (10 points)

$$\rho_{tt} - c^2\Delta\rho = -\rho_0 \operatorname{div}\vec{F} \qquad c^2 = p'(\rho_0).$$