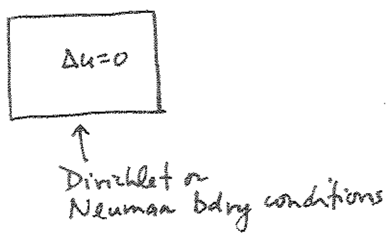
Math 5440
Friday 10/29

Hw for Friday 11/5
4.23.2, 4.23.3, 4.23.7
4.24.1, 4.24.8
more on Monday;
class exercise(s)?

- Finish Wednesday notes, beginning with the important analysis theorem which justifies differentiating infinite sums term by term.

  Hint: Show f is an antiderivative of h, by computing $\int_a^x h(\tilde{x})\,d\tilde{x}$.

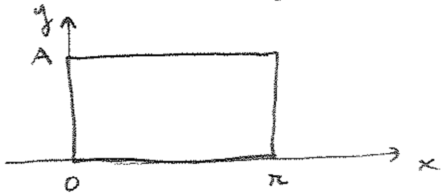
- §4.23 Laplace in a rectangle.



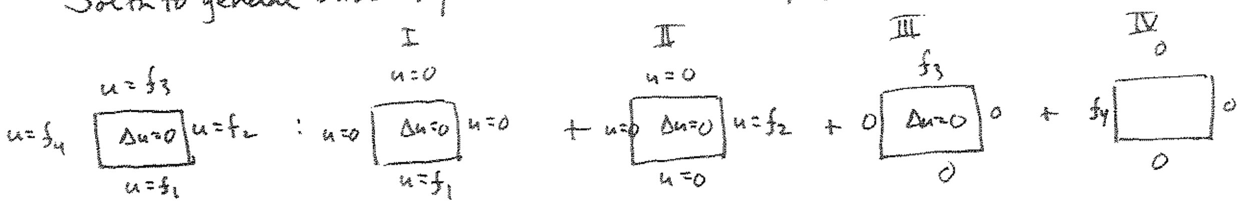
<u>scaling</u> if $u(x,y) = \tilde{u}(ax, by)$
then $u_{xx} = a^2 \tilde{u}_{\tilde{x}\tilde{x}}$
$u_{yy} = b^2 \tilde{u}_{\tilde{y}\tilde{y}}$

so if $a=b$ (uniform scaling), then $\Delta u = 0$ iff $\Delta \tilde{u} = 0$.

So we (after also doing a translation) assume our coord. rectangle is

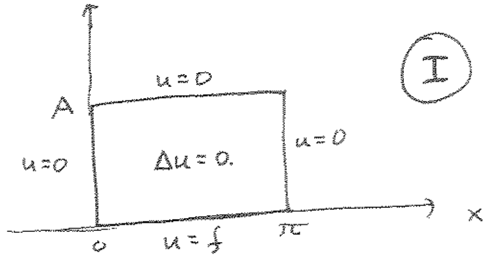


Soltn to general Dirichlet problem for $\Delta u = 0$ is superposition of 4 other soltns:



(analogous for mixed Dirichlet/Neumann problems).

focus on this one



(I)

separated sol't'ns for $u_{xx}+u_{yy}=0$

$$u(x,y)=X(x)Y(y)$$

$$\Delta u = X''Y+XY''=0$$

$$-\frac{X''}{X}=\frac{Y''}{Y}=\lambda \quad \text{const}$$

for our Dirichlet Problem soltns

$$-\frac{X''}{X}=\lambda \implies X_n(x)=b_n\sin nx \qquad n\in\mathbb{N}, \text{ as before.}$$

$$X(0)=X(\pi)=0 \qquad \lambda_n=n^2$$

$\Rightarrow Y_n(y)$ satisfies

$$\frac{Y''}{Y}=n^2 \qquad\qquad Y''-n^2Y=0$$

$$Y=c_1e^{ny}+c_2e^{-ny}$$

$Y(0)=1$ (normalized)

$Y(A)=0$

$$Y(0)=1,\ Y(A)=0 \implies Y_n(y)=\frac{\sinh(n(A-y))}{\sinh(nA)}$$

$$X_n(x)Y_n(y)=b_n\sin nx\left(\frac{\sinh(n(A-y))}{\sinh nA}\right).$$

formal soltn to (I): Extend f as 2π-periodic odd fcn.

Fourier $\quad f\sim\sum_{n=1}^{\infty}b_n\sin nx \qquad b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx.$

solt?: $$\boxed{u(x,y)=\sum_{n=1}^{\infty}b_n\sin nx\left(\frac{\sinh(n(A-y))}{\sinh nA}\right)}$$

$\Delta u=0$? for $A>y\geq\delta>0$ terms converge absolutely, exponentially fast, even after taking arbitrarily many x or y derivs, just as for heat eqtn soltn.
So, for $A>y>0$ u is ∞'ly differentiable, and each partial deriv of arbitrary order, can be computed term by term

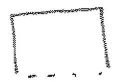
$$\therefore\ u_{xx}+u_{yy}=\sum_{n=1}^{\infty}(\partial_x^2+\partial_y^2)(n^{\text{th}}\text{ term})=0.$$

$$\frac{\sinh(n(A-y))}{\sinh nA}=\frac{\frac12\left(e^{n(A-y)}-e^{n(y-A)}\right)}{\frac12\left(e^{nA}-e^{-nA}\right)}=\frac{\cancel{e^{nA}}}{\cancel{e^{nA}}}\left(\frac{e^{-ny}-e^{n(y-2A)}}{1-e^{-2nA}}\right)$$

decays exponentially fast for $0<\delta\leq y\leq 2A-\delta$.

F.S.

soln to I solves Dirichlet boundary values?

$u = 0$ on  by construction

and series converges uniformly in closed rectangle, for $y \geq \delta$
so extends continuously to the closed rectangle.

whether $u(x,0) = f(x) \quad 0 \leq x \leq \pi$
and whether u extends continuously to the closed rectangle is same question as for heat equation, and has analogous answers.

Example theorem: If $f(x)$ is piecewise C^1 on $[0,\pi]$ and continuous, with $f(0) = f(\pi)$, then

$$u(x,y) = \sum_{n=1}^{\infty} \underbrace{b_n \sin nx \left(\frac{\sinh n(A-y)}{\sinh nA} \right)}_{\%_n}$$

- is continuous on $[0,\pi] \times [0,A]$
- is C^∞ in $(0,\pi) \times (0,A)$

and solves the Dirichlet problem I.

In fact, the partial sums
$$S_N(u;(x,y)) = \sum_{n=1}^{N} \%_n$$ converge uniformly in $u(x,y)$ in $[0,\pi]\times[0,A]$.

proof: $S_N(f;x) \to f(x)$ uniformly on $[0,\pi]$.

By maximum principle, for $(x,y) \in [0,\pi] \times [0,A]$,
$$|S_N(u;(x,y)) - S_M(u;(x,y))| \leq \max \text{ on bdry } = \max |S_N(f;x) - S_M(f;x)|$$

(since the partial sums are harmonic functions which are continuous on the closed domain)

So $\{S_N\}$ is uniformly Cauchy on $[0,\pi]\times[0,A]$
so it converges to a continuous limit, uniformly.

Explicit estimate for $|S_N(u) - u|$:

(as in example page 97)

for $\int_0^\pi (f')^2 < \infty$

$$|S_N(u;(x,y)) - u(x,y)| \le \max_{0\le x\le \pi} |S_N(f;x) - f(x)|$$

$$|S_N(f;x) - f(x)| = \Big|\sum_{n>N}{}' b_n \sin nx\Big|$$

$$\le \sum_{n>N}{}' |b_n| = \sum_{n>N} n|b_n| \tfrac{1}{n}$$

$$\le \Big(\sum_{n>N} n^2 b_n^2\Big)^{1/2} \Big(\sum_{n>N} \frac{1}{n^2}\Big)^{1/2}$$

$$\Big(\frac{\pi^2}{6} - \sum_{n=1}^{N} \frac{1}{n^2}\Big)^{1/2}$$

$$\Big(\frac{2}{\pi}\Big[\int_0^\pi f'(x)^2 dx - \sum_{n=1}^{N} n^2 b_n^2\Big]^{1/2}\Big)$$

$$f = \sum_{n=1}^{\infty} b_n \sin nx$$

$$f' \sim \sum_{n=1}^{\infty} n b_n \cos nx$$

Parseval. $\|f'\|^2_{[-\pi,\pi]} = \sum_{n=1}^{\infty}{}' n^2 b_n^2$

Pythag.: $$\sum_{n>N} n^2 b_n^2 = \underbrace{\frac{2}{\pi}\int_0^\pi f'(x)^2 dx}_{\|f'\|^2} - \underbrace{\sum_{n=1}^{N} n^2 b_n^2}_{\|S_N(f';x)\|^2}$$