Math 5440
Wed 10/27

Using convergence theorems to check that our earlier formal solns to IBVP's for PDE's are really solns.

Summary of convergence theorems for Fourier series for 2π-periodic functions

A) pointwise convergence : $\int_{-\pi}^{\pi} |f(x)|\,dx < \infty$, local Lipschitz or Hölder estimate for continuity of f at x_0 $\Rightarrow S_N(f; x_0) \to x_0$.

B) uniform convergence : f continuous, $\int_{-\pi}^{\pi} f'(x)^2\,dx < \infty \Rightarrow S_N(f) \to f$ uniformly.

$$\Downarrow$$

$$\sum_{-\infty}^{\infty} n^2 |c_n|^2 < \infty$$

$\Rightarrow$ uniform conv. of $S_N(f)$ to a limit fcn.

$$|f(x_2) - f(x_1)| \le \left|\int_{x_1}^{x_2} |f'(x)|\,dx\right| \le \left(\int_{x_1}^{x_2} |f'(x)|^2\right)^{1/2} \underbrace{\left(\int_{x_1}^{x_2} 1^2\,dx\right)^{1/2}}_{|x_2 - x_1|^{1/2}}$$

Hölder ½

So pointwise convergence $S_N(f; x) \to f(x)\ \forall x$.

C) $S_N(f) \to f$ in $\|\cdot\|_{L^2}$

$\forall f \in V = \{ f : \mathbb{R} \to \mathbb{R}$ s.t. f 2π-periodic, bd, piecewise continuous$\}$

(actually true $\forall f$ 2π periodic s.t. $\underbrace{\int_{-\pi}^{\pi} f^2(x)\,dx < \infty}_{f \in L^2(-\pi,\pi)}$).

proof of B $\Rightarrow$ C relies on approximation lemma

<u>Lemma</u> Let $f \in V$, $\varepsilon > 0$. Then $\exists$ piecewise linear continuous g s.t. $\|f - g\|_{L^2} < \varepsilon/2$

(actually true for $f \in L^2(-\pi,\pi)$).

assuming lemma,

Let $\varepsilon > 0$. Find g. From (B) $\exists$ M s.t. $N \ge M \Rightarrow \|g - S_N(g)\|_{L^2} < \varepsilon/2$

thus for $N \ge M$
$$\|f - S_N(f)\|_{L^2} \le \|f - S_N(g)\|_{L^2} \le \|f - g\|_{L^2} + \|g - S_N(g)\|_{L^2} < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \blacksquare$$

nearest fcn to f is V_N in our inner product space

Armed with these convergence theorems,
our task in the rest of Chapter 4 is to show that our formal separated-variable Fourier solns to the Laplace, wave, heat eqns BVP's & IBVP's actually give real solutions.

Key analysis lemma for interchanging ∞ sums and derivatives:

<u>Theorem</u> Let $\{f_j\}$ be a seq. of cont. fcns on $[a,b] \subset \mathbb{R}$ with $f_j, f_j' \in C[a,b]$, i.e. $f_j \in C^1[a,b]$, $\forall j$.

Let $\sum_{j=1}^{N} f_j \to f$ uniformly

$\sum_{j=1}^{N} f_j' \to h$ uniformly.

Then $f \in C^1([a,b])$ and $f' = h$.

This is one of those basic results every good mathematician should be able to prove:

a) f is continuous (uniform limits of cont. fcns. are cont.)

b) $f' = \sum f_j'$

Let's try!

§ 4.22 Heat eqtn revisited.

$$\begin{cases} u_t - k u_{xx} = 0 & 0<x<\pi,\ t>0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

(homogeneous) Dirichlet IBVP

$$X_n T_n = (\sin nx)\, e^{-kn^2 t}$$

formal soltn: extend f as odd 2π-periodic

$$f \sim \underbrace{\frac{a_0}{2}}_{0} + \underbrace{\sum_{n=1}^{\infty} \cancel{a_n \cos nx}}_{0} + \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(\bar{x}) \sin n\bar{x}\, dx$$

formal soltn

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx\, e^{-kn^2 t}$$

$$\begin{cases} u_t - k u_{xx} = 0 & 0<x<\pi,\ t>0 \\ u_x(0,t) = u_x(\pi,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$X_n T_n = (\cos nx)\, e^{-kn^2 t}$$

extend f as even 2π-periodic fcn

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \underbrace{\cancel{\sum_{n=1}^{\infty} b_n \sin nx}}_{0}$$

$$\frac{a_0}{2} = \frac{1}{\pi}\int_0^{\pi} f(\bar{x})\,d\bar{x}\ ; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(\bar{x}) \sin n\bar{x}\, d\bar{x}$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\, e^{-kn^2 t}$$

- Do formal soltns satisfy $u_t - k u_{xx} = 0$?

  $f \in L^1 \Rightarrow \{a_n\}, \{b_n\} \to 0.$

  $\forall M \in \mathbb{N}$, $\sum_{n=1}^{\infty} n^M e^{-kn^2 t}$ is uniformly convergent for $t \geq \delta > 0$

  $\therefore$ $u(x,t)$ is ∞'ly differentiable in (x,t), and all partial derivatives of $u(x,t)$ are the sums of corresponding partials

  $$\Rightarrow (\partial_t - k\partial_x^2)\, u = \sum_{n=1}^{\infty} b_n (\partial_t - k\partial_x^2)(X_n T_n) = 0 \quad \blacksquare$$

- Dirichlet (or Neumann) boundary conditions are satisfied by construction.

- In what sense does $u(x,0) = f(x)$? This depends on the regularity of f.

- If $f \in L^2([0,\pi])$ (or $[-\pi,\pi]$ after extension)

  then $S_N(f) \to f$ in $L^2([-\pi,\pi])$ (so also in $[0,\pi]$)

  and $\|u(\cdot,t)-f(\cdot)\|_{L^2}^2 = \sum_{n=1}^{\infty}\left(b_n(e^{-knt}-1)\right)^2 \le \left(e^{-kt}-1\right)\sum_{n=1}^{\infty} b_n^2 \to 0$ as $t \to 0$.

  Parseval! (or a_n, for Neumann)

Stronger convergence:

- If $\int_0^\pi f'(x)^2\,dx < \infty$ (and if, for Dirichlet problem, $f(0)=f(\pi)=0$)

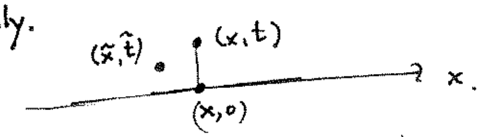
  then the 2π-periodic extension of f is Hölder ½ continuous,

  and $S_N(f,x) \to f(x)$ uniformly, (convergence theorems A, B).

  so $\underline{u(x,0)=f(x)}$ $0 \le x \le \pi$.

Furthermore u extends to be continuous, $u \in C([0,\pi]\times[0,\infty))$:

Step 1: $\lim_{t\to 0^+} u(x,t) = f(x)$ uniformly.



proof:

$$u(x,t)-f(x) = \sum_{n=1}^{\infty} b_n(e^{-knt}-1)\sin nx$$

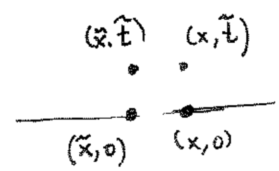
$$= \sum_{n=1}^{N} b_n(e^{-knt}-1)\sin nx + \underbrace{\sum_{n=N+1}^{\infty} b_n(e^{-knt}-1)\sin nx}_{\le \sum_{N+1}^{\infty}|b_n| \le \left(\sum_{N+1}^{\infty}(nb_n)^2\right)^{1/2}\left(\sum_{N+1}^{\infty}\frac{1}{n^2}\right)^{1/2}}$$

$\to 0$ as $N \to \infty$.

Let $\varepsilon > 0$. Pick M s.t.
$N \ge M \Rightarrow |\text{term2}| < \varepsilon/2$.

then pick δ s.t. $|t| < \delta$
$\Rightarrow |1^{st}\text{ term}| < \varepsilon/2$ ∎

Step 2 $\lim_{(\tilde{x},\tilde{t})\to(x,0)} u(\tilde{x},\tilde{t}) = f(x)$



proof $u(\tilde{x},\tilde{t})-f(x)$

$$= \underbrace{u(\tilde{x},\tilde{t})-f(\tilde{x})}_{\substack{\text{small for}\\ \tilde{t}\text{ small}}} + \underbrace{f(\tilde{x})-f(x)}_{\substack{\text{small for}\\ |\tilde{x}-x|\text{ small}}}$$ ∎