Math 5440
Friday Oct 1

$\boxed{\text{Exam Monday in } \underline{\text{JWB 240}} \text{ come for any 80 consecutive minutes between 10:15 - 12:00}}$

1-space dim
Heat equation: $u_t - ku_{xx} = 0$

Example: The function $Q(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}}$ solves $u_t - ku_{xx} = 0$ (except at singular point $(0,0)$)

It's called "the fundamental sol'n" of the one space variable heat eqtn

```
> Q := (x, t, k) -> 1/(2·sqrt(π·k·t)) · exp(-x^2/(4·k·t)); #Heat source
```

$$Q := (x, t, k) \to \frac{1}{2} \frac{e^{-\frac{1}{4}\frac{x^2}{k t}}}{\sqrt{\pi k t}}$$

```
> diff(Q(x, t, k), t) - k·diff(Q(x, t, k), x, x) : #verify solution to heat equation
  simplify(%);
```

$$0$$

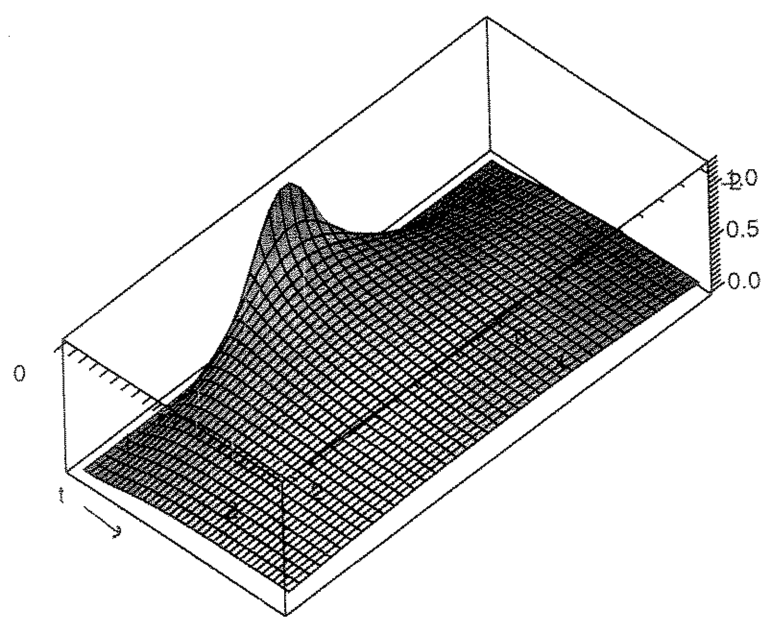
"just" a calculation to check it solves heat eqtn. better question is, where did this soltn come from?

```
> int(Q(x, t, k), x = -∞ .. ∞); #heat is conserved
```

$$\begin{cases} \frac{1}{\sqrt{k t}\sqrt{\frac{1}{k t}}} & \mathrm{csgn}\left(\frac{1}{k t}\right) = 1 \quad t>0 \\ \infty & \text{otherwise} \quad t<0 ? \end{cases}$$

for $t>0$, $\int_{-\infty}^{\infty} Q(x,t)\,dx = 1$
conservation of energy!

```
> with(plots) :
  plot3d(Q(x, t, 1), x = -2 .. 2, t = 0 .. 2, grid = [40, 40], axes = boxed, scaling = constrained);
```



$Q(x,t)$ is the solution to the generalized Dirichlet problem in the upper half plane

$$\begin{cases} u_t - ku_{xx} = 0 & x \in \mathbb{R},\ t>0 \\ u(x,0) = \delta(x) & x \in \mathbb{R} \\ |u(x,t)| \le M & \end{cases}$$

the δ "function" (really a measure) based at the origin.

- Notice the domain of dependence of the origin is the entire upperhalf plane! Information propogates instantaneously with ∞ speed.

The solution for

$$\begin{cases} u_t - k u_{xx} = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \end{cases}$$

for f continuous and bounded on $\mathbb{R}$ and $u(x,t)$ bounded on $\mathbb{R} \times \mathbb{R}^+$ is given by a convolution (averaging) process with the fundamental sol'n (this convolution is more common than the related Laplace transform convolution of 2280)

$$u(x,t) = \int_{-\infty}^{\infty} f(\alpha)\, Q(x-\alpha, t)\, d\alpha \qquad (x,t),\ x \in \mathbb{R},\ t > 0$$

- If you can justify passing the operator $\partial_t - k\partial_x^2$ through the integral sign, $u(x,t)$ is harmonic.
- for t small and x near α you are averaging values of f with most of the weight of the average occuring very close to α, so $u(x,t)$ will be close to $f(\alpha)$
  → making this precise proves that $u(x,t)$ extends continuously to the real axis ($t=0$), with boundary values f.

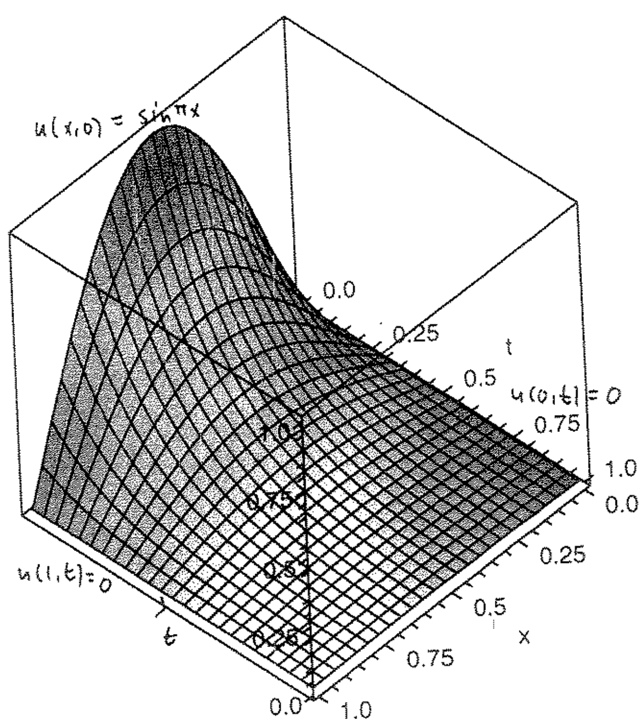
```
> heatconvolution := (f, k, N, x, t) → int(f(α)·Q(x − α, t, k), α = −N..N);
```

$$heatconvolution := (f, k, N, x, t) \rightarrow \int_{-N}^{N} f(\alpha)\, Q(x-\alpha, t, k)\, d\alpha$$

```
> g := α→sin(π·α); #odd with respect to alpha = 0 and 1
```

$$g := \alpha \rightarrow \sin(\pi\,\alpha)$$

```
> heatconvolution(g, k, N, x, t) :
> plot3d(heatconvolution(g, 0.5, 10.0, x, t), x = 0..1, t = 0..1, axes = boxed, scaling
     = constrained);
```



$$\begin{cases} u_t - .5\, u_{xx} = 0 \\ u(x,0) = \sin \pi x \leftarrow \text{odd @ } x=0,\ x=1 \end{cases}$$

sol'n is odd extension of function graphed at right (odd wrt $x=0$, $x=1$)

Also solves

$$\text{IBVP} \begin{cases} u_t - .5\, u_{xx} = 0 \\ u(x,0) = \sin \pi x & 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0 & t > 0 \end{cases}$$

So, because the Gaussian averaging is symmetric (even), you can ~~so~~ use even/odd reflections of initial data to solve Dirichlet/Neumann type IBVP's, just as for wave equation.

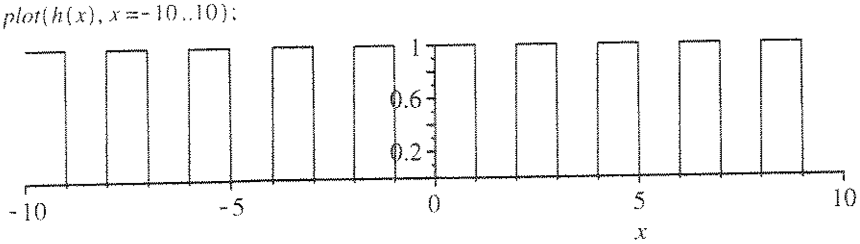
Below is solution graph for

$$\begin{cases} u_t - .5u_{xx} = 0 & 0<x<1,\ t>0 \\ u(x,0) = 1 & 0<x<1 \\ u(0,t) = u(1,t) = 0 & t>0 \end{cases}$$

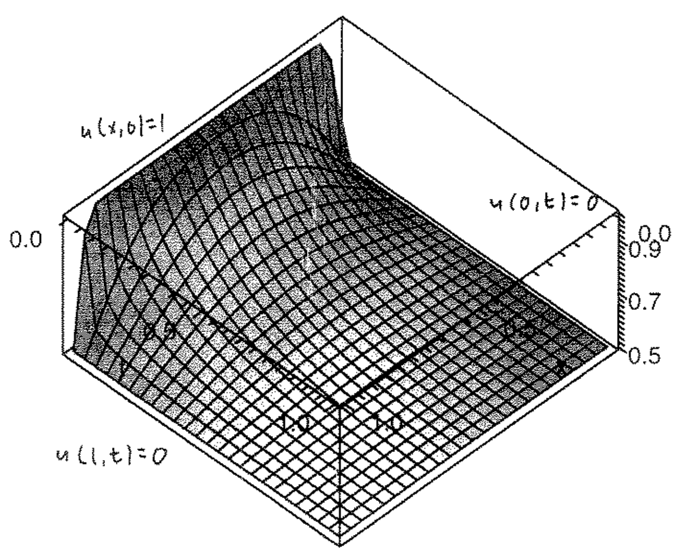
> $h := x \rightarrow sum((-1)^j \cdot \text{Heaviside}(x-j), j=-10..10)$; *#odd extension of the function #"1", with respect to x=0 and x=1*

$$h := x \rightarrow \sum_{j=-10}^{10} (-1)^j \,\text{Heaviside}(x-j)$$

> *plot*$(h(x), x=-10..10)$;



> *plot3d*(*heatconvolution*(*h*, 0.5, 10.0, *x*, *t*), *x* = 0..1, *t* = (0.001)..1, *axes = boxed, scaling = constrained*);



- How would you solve the Neuman IBVP:

$$\begin{cases} u_t - ku_{xx} = 0 & 0<x<1 \quad t>0 \\ u(x,0) = 1 & 0<x<1 \\ u_x(0,t) = u_x(1,t) = 0 & t>0 \end{cases}$$

Exam 1 discussion

- you'll have 1:20 of time, between 10:15 & 12:50 non, in JWB 240 (by main office)

- I expect the exam will of the form "do 3 of the following 4 problems" (or 4 of 5, or 4 of 6). problems may have multiple parts.

- in my mind, the goal of exams like this is to give you an opportunity to distill key concepts, techniques, & results, and to demonstrate your mastery to both me and you.

the tools you should know and be able to use include
chain rule, div, grad, curl, COV, vector calc thms (FTC, Green's, Stokes', div)
$\uparrow$ for derivatives & integrals

you should understand PDE derivations we went over in class and on HW

you should understand the integral & differential calc techniques for $\exists, !$ to BVP's / IBVP's

you should understand HW — not specific problems necessarily, but the techniques which enable you to solve those & similar problems (By this I mean, it could help you some, but likely not a substantial amount, to memorize sol'ns to sample problems; but, by all means do review your HW as well as classnotes).

Ask questions now about concepts, particular areas you want clarified in terms of your responsibility!