Math 5440
Wed 11/3

Monday notes, except:

... proof of page 3

<u>Theorem</u>: If f is continuous on ∂D, then the harmonic function given in polar coords (r,θ) $0 \le r < 1$

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta-\varphi)\left(\frac{1-r^2}{(1-r\cos\varphi)^2+(r\sin\varphi)^2}\right)d\varphi$$

extends continuously to the closed unit disk $\{r \le 1\}$ with boundary values f

<u>proof</u>: we use the approximate identity properties for $P(r,\varphi) = \dfrac{1-r^2}{(1-r\cos\varphi)^2+(r\sin\varphi)^2}$

a) $\displaystyle\int_{-\pi}^{\pi} P(r,\varphi)\,d\varphi = 2\pi \qquad r<1$

b) $P(r,\varphi) > 0 \qquad r<1$

c) for $|\varphi| \ge \delta > 0$, $\quad 0 \le P(r,\varphi) \le \dfrac{1-r^2}{(\delta\sin\varphi)^2} \to 0$ uniformly as $r \to 1^-$

<u>Step 1</u> $\quad u(r,\theta) \to f(\theta)$ uniformly in θ as $r \nearrow 1$.

$$f(\theta) - u(r,\theta) = f(\theta) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta-\varphi)\left(P(r,\varphi)\right)d\varphi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(\theta)-f(\theta-\varphi)\right)P(r,\varphi)\,d\varphi \qquad \text{(A).}$$

$$= \frac{1}{2\pi}\int_{|\varphi|<\delta}\left(f(\theta)-f(\theta-\varphi)\right)P(r,\varphi)\,d\varphi + \frac{1}{2\pi}\int_{\delta<|\varphi|<\pi}\left(f(\theta)-f(\theta-\varphi)\right)P(r,\varphi)\,d\varphi.$$

$$= I_1 + I_2.$$

Let $\varepsilon>0$. <u>Pick δ</u> s.t. $|\varphi|<\delta \Rightarrow |f(\theta)-f(\theta-\varphi)| < \varepsilon\pi \;\; \forall\theta$ (f unif. cont.).

$$\Rightarrow |I_1| \le \frac{1}{2\pi}\int_{|\varphi|<\delta} \underbrace{|f(\theta)-f(\theta-\varphi)|}_{<\varepsilon\pi} P(r,\varphi)\,d\varphi \qquad \text{(A.B)}$$

$$< \frac{\varepsilon}{2}\int_{|\varphi|<\delta} P(r,\varphi)\,d\varphi < \varepsilon/2.$$

Also $|f| \le M$ some $M<\infty$, so $|f(\theta-\varphi)-f(\theta)| < 2\pi$.

Pick δ_1 s.t. $\quad 1-\delta_1 < r < 1 \Rightarrow P(r,\varphi) < \dfrac{\varepsilon}{4M}$

$$\Rightarrow I_2 \le \frac{1}{2\pi}\int_{\delta<|\varphi|<\pi} 2M\cdot\frac{\varepsilon}{4M}\,d\varphi < \frac{1}{2\pi}\,\frac{\varepsilon}{2}\cdot 2\pi = \varepsilon/2$$

So for $1-\delta_1<r<1$, $\quad |f(\theta)-u(r,\theta)| < \varepsilon$ ∎

<u>Step 2</u>. $\lim\limits_{(r,\bar\theta)\to(1,\theta)} u(r,\bar\theta) = f(\theta)$

(for $r=1$ define $u(r,\theta)=f(\theta)$).

$$u(r,\bar\theta) - f(\theta) = u(r,\bar\theta) - f(\bar\theta) + f(\bar\theta) - f(\theta).$$

$$|u(r,\bar\theta) - f(\theta)| \le |u(r,\bar\theta) - f(\bar\theta)| + |f(\bar\theta) - f(\theta)|.$$

Let $\varepsilon > 0$. pick δ s.t. $|\theta - \bar\theta| < \delta \Rightarrow |f(\bar\theta) - f(\theta)| < \varepsilon/2$
pick δ_1 s.t. $1-\delta_1 < r \le 1 \Rightarrow |u(r,\bar\theta) - f(\bar\theta)| < \varepsilon/2 \quad \forall \bar\theta$

so $|\theta - \bar\theta| < \delta$ and $1-\delta_1 < r \le 1$ $\Rightarrow |u(r,\bar\theta) - f(\theta)| < \varepsilon$ ∎

What's next?

- non-~~homogeneous~~ IBVP's for H.E. & W.E. $\quad u_t - k\Delta u = F$, $\quad u_{tt} - c^2\Delta u = F$ $\quad$ 1 space dim
  (just turns into solvable O.D.E's for x-Fourier coeffs of $u(x,t)$)

- higher space dimensions ??
  What takes the place of $\{\sin nx\}_{n\in\mathbb{N}}$, $\{\cos nx\}_{n\in\mathbb{Z}^+}$??
  Dirichlet $\quad$ Neumann

  It's the Spectral Theorem, folks.

<u>Question</u>: if $\Delta u = 0$ in disk of radius a centered at origin $B(0;a)$
$u = f$ on ∂D

what is the Poisson integral formula for f?
Hint: scale!