HW will be due Wed next week. Do you want a Tues or Mond. problem session?

Math 5440
Wed 11/17

Chapter 7 of Strauss text: Green's functions and divergence theorem magic for Laplace's equation

<u>Note</u>: Strauss is working in $\mathbb{R}^3$. We will include $\mathbb{R}^2$ and $\mathbb{R}^n$ $n>3$, at the same time.

- Recall these FTC's for bounded, piecewise smooth domains $\Omega \subset \mathbb{R}^n$

(i) $\displaystyle\int_\Omega \frac{\partial u}{\partial x_i}\, dV_n = \int_{\partial\Omega} u\, \hat{e}_i \cdot \hat{n}\, dV_{n-1}$

(ii) $\displaystyle\int_\Omega \vec{\nabla}\cdot\vec{F}\, dV_n = \int_{\partial\Omega} \vec{F}\cdot\hat{n}\, dV_{n-1}$

Strauss notation $\nabla v\cdot\hat{n} \;``="\; \frac{\partial v}{\partial n}$ (direc. deriv in $\hat{n}$ dir.)

(iii) $\displaystyle\int_\Omega u\Delta v + \nabla u\cdot\nabla v\, dV_n = \int_{\partial\Omega} u\nabla v\cdot\hat{n}\, dV_{n-1}$ ← with $u=v$ we used this to show uniqueness for D.P. for Δ equation; uniqueness up to additive const for N.P.

(iv) $\displaystyle\int_\Omega u\Delta v - v\Delta u\, dV_n = \int_{\partial\Omega} u\nabla v\cdot\hat{n} - v\nabla u\cdot\hat{n}\, dV_{n-1}$

- Now consider the following rotationally symmetric <u>fundamental</u> <u>sol</u>t<u>ns</u> for Laplace operator

$K(x) = \frac{1}{2}|x| \quad n=1$

$K(\vec{x}) = -\frac{1}{2\pi}\ln|\vec{x}| \quad n=2$

$K(\vec{x}) = -\frac{1}{4\pi}\frac{1}{|\vec{x}|} \quad n=3$

$K(\vec{x}) = -\frac{1}{c_n}|\vec{x}|^{2-n} \quad n\geq 3$ s.t. $\frac{c_n}{n-2}$ = surface area (volume) of unit sphere $\{\vec{x} \mid |\vec{x}|=1\}$ in $\mathbb{R}^n$ ↑ $S^{n-1}(0)$ ↑ center

These functions have the property that

(i) $\Delta K = 0$ for $\vec{x}\neq\vec{0}$ ← we've checked for $n=1,2,3$.

(ii) $\forall \varepsilon>0$, $\displaystyle\int_{S^{n-1}_\varepsilon(0)} \nabla K\cdot\hat{n}\, dV_{n-1} = 1$ $\left(``="\displaystyle\int_{B^n_1(0)} \Delta K\, dV_n\right)$ ← check this!!

↑ radius

i.e. $\Delta K \;``="\; \delta(\vec{x})$.

- Why would it make sense to look for a rotationally symmetric fundamental soltns?

- Notice $K(\vec{x}-\vec{x}_0)$ and $K(\vec{x}-\vec{x}_0) + \underset{\text{harmonic}}{u(\vec{x})}$ will have analogous properties the magic begins...

<u>Def.</u> Denote the average value of a function over balls & spheres as follows:

$$\fint_{S^{n-1}_R(x_0)} u(x)\,dV_{n-1} := \frac{1}{|S^{n-1}_1|R^{n-1}} \int_{S^{n-1}_R(x_0)} u(x)\,dV_{n-1}$$

$$\fint_{B^n_R(x_0)} u(x)\,dV_n = \frac{1}{|B^n_1|R^n} \int_{B^n_R(x_0)} u(x)\,dV_n$$

<u>Theorem 1</u> (mean value property) Let $u \in C^2(\Omega)$ be harmonic, $\Delta u = 0$.
Let $\overline{B_R(x_0)} \subset \Omega$.
Then
$$u(x_0) = \fint_{S^{n-1}_R(x_0)} u(x)\,dV_{n-1} = \fint_{B_R(x_0)} u(x)\,dV_n.$$

proof: Let $0 < \varepsilon < R$.
Let $v(x) = K(x - x_0) - K(R)$.

$$0 = \int_{B_R(x_0)\setminus B_\varepsilon(x_0)} u \overset{=0}{\Delta v} - v \overset{=0}{\Delta u} = \int_{S_R(x_0)} u\frac{\partial v}{\partial n} - \overset{=0}{v}\frac{\partial u}{\partial n} - \int_{S_\varepsilon(x_0)} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}$$

$$= I_1 - I_2.$$

on any sphere $S_r(x_0)$, $\frac{\partial v}{\partial n} = \frac{\partial v}{\partial r} = d_n r^{1-n}$ with $d_n = \frac{1}{|S^{n-1}_1|}$

Thus
$$I_1 = \frac{1}{|S^{n-1}_1|R^{n-1}} \int_{S_R(x_0)} u(x)\,dx = \fint_{S_R(x_0)} u(x)\,dx.$$

$$-I_2 = -\fint_{S_\varepsilon(x_0)} u\,dx + \underbrace{\int_{S_\varepsilon(x_0)} v\frac{\partial u}{\partial n}}_{|\ |\le c\varepsilon^{n-1}\varepsilon^{2-n}}$$

(or $|\ln\varepsilon|$ $n=2$)

$\to -u(x_0)$ as $\varepsilon \to 0$. $\qquad \to 0$ as $\varepsilon \to 0$

As $\varepsilon \to 0$, deduce

$$0 = \fint_{S_R(x_0)} u(x)\,dx - u(x_0)$$

∎

(integral averages for balls in HW)

<u>Theorem 2</u> Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfy $\Delta u = 0$. Let $x_0 \in \Omega$.
Then for $v(x) = K(x - x_0)$,

$$u(x_0) = \int_{\partial\Omega} u(x)\frac{\partial v}{\partial n} - v(x)\frac{\partial u}{\partial n}\, dV_{n-1}$$

proof: this is the same proof as in Theorem 1, but applied to $\Omega \setminus B_\varepsilon(x_0)$, as $\varepsilon \to 0$.

(not so useful if you want to solve Dirichlet problem, because you'd also need Neumann data (& vise verse))

<u>Def</u> (using Strauss' notation). The Green's function $G(x, x_0)$, $G: \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$, for the Dirichlet problem and the operator $-\Delta$ satisfies

(i) $G(x, x_0) \in C^2(\Omega \setminus \{x_0\}) \cap C^1(\bar{\Omega} \setminus \{x_0\})$, $\Delta_x G(x, x_0) = 0$

(ii) $G(x, x_0) = 0 \quad \forall x \in \partial\Omega$

(iii) $G(x, x_0) - K(x - x_0) \in C^2(\Omega)$ and is harmonic.

<u>Theorem</u> Green's functions are unique, symmetric, and they exist

not hard

hard.

<u>Theorem 3</u> Let $G(x, x_0)$ be the Green's function above. Let u solve

$$\Delta u = 0 \quad \text{in } \Omega$$
$$u = f \quad \text{on } \partial\Omega$$

Then for $x_0 \in \Omega$,

$$u(x_0) = \int_{\partial\Omega} u(x)\frac{\partial G(x, x_0)}{\partial n}\, dV_{n-1}$$

Generalizes Poisson integral formula to any domain

$$\left(u(x) = \int_{\partial\Omega} u(\xi)\, \nabla_\xi(G(\xi, x)) \cdot \hat{n}\, dV_{n-1} \right)$$

<u>Proof</u>: $G(x, x_0) = K(x - x_0) + w(x)$, $w(x)$ harmonic.
Use ~~result~~ of Theorem 2 method.