Math 5440

Friday Nov. 12

5.27 – 5.28 ← Green's fcns for 2nd order DE BVP's

↑ variation of parameters for 2nd order DE IVP

5.27 to solve

$$* \begin{cases} \frac{d}{dx}\left(p(x)u'\right) + qu = f & \alpha < x < \beta \\ u(\alpha) = 0 \\ u'(\alpha) = 0 \end{cases}$$

$pu'' + p'u' + qu = f$

$u'' = -\frac{p'}{p}u' - \frac{q}{p}u + \frac{f}{p}$

from a basis $\{v_1(x), v_2(x)\}$ for sol'ns to homogeneous DE $(pu')' + qu = 0$.

- Recall variation of parameters for 1st order systems of DE's

$$\vec{x}'(t) = \underset{\substack{\uparrow \\ n\times n \text{ matrix}}}{A}\,\vec{x}(t) + \vec{F}$$

$n\times n$ ~~const.~~ matrix, $A = A(t)$. (or const.).

Let $\Phi(t) = [y_1 | y_2 | \cdots y_n]$ be a FMS; $\Phi' = A\Phi$ $\det \Phi \neq 0$.

Search for

$$\vec{x}(t) = \Phi(t)\,\underset{\substack{\uparrow \\ \text{"varying parameters"}}}{\vec{c}(t)}$$

$$x' = \cancel{\underset{\substack{\uparrow \\ A\Phi}}{\Phi' c}} + \Phi c' \underset{\substack{\uparrow \\ \text{want}}}{=} \cancel{A\Phi \vec{c}} + \vec{F}$$

iff $\vec{c}' = \Phi^{-1}\vec{F}$

e.g. $\vec{c}(t) = \int_\alpha^t \Phi^{-1}(s)\vec{F}(s)\,ds$; $\vec{x}(t) = \underbrace{\Phi(t)\vec{c}(t)}_{\vec{x}_P} + \underbrace{\Phi(t)\vec{d}}_{\vec{x}_H}$

converting variable x to variable t,

$*$ is equivalent to 1st order system for $\begin{bmatrix} u(t) \\ u'(t) \end{bmatrix}$: $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -q/p & -p'/p \end{bmatrix} + \begin{bmatrix} 0 \\ f/p \end{bmatrix}$

$$\Phi(t) = \begin{bmatrix} v_1(t) & v_2 \\ v_1'(t) & v_2' \end{bmatrix}$$

So $\begin{bmatrix} u \\ u' \end{bmatrix} = \Phi(t)\vec{c}(t) + \vec{x}_H$

$$\begin{bmatrix} u(t) \\ u'(t) \end{bmatrix} = \Phi(t)\int_\alpha^t \underbrace{\frac{1}{W(s)}\begin{bmatrix} v_2'(s) & -v_2 \\ -v_1' & v_1 \end{bmatrix}\begin{bmatrix} 0 \\ \frac{f(s)}{p(s)} \end{bmatrix}}_{\frac{f}{pW}\begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}} ds = \int_\alpha^t \Phi(t)\frac{f(s)}{(pW)(s)}\begin{bmatrix} -v_2(s) \\ v_1(s) \end{bmatrix} ds + \vec{x}_H$$

$\begin{bmatrix} v_1 & v_2 \\ v_1' & v_2' \end{bmatrix}$

$p(s)W(s)$ is constant!

$$(pW)' = \left(p\begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix}\right)'$$

$$= p'\begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix} + p\begin{vmatrix} v_1' & v_2 \\ v_1'' & v_2' \end{vmatrix} + p\begin{vmatrix} v_1 & v_2' \\ v_1' & v_2'' \end{vmatrix}$$

$$v_1'' = -\frac{p'}{p}v_1' - \frac{q}{p}v_1 \qquad v_2'' = -\frac{p'}{p}v_2' - \frac{q}{p}v_2$$

$= 0$!

So (going back to x variable), and looking at 1st entry we get

$$\boxed{u(x) = \int_\alpha^x R(x,\xi) f(\xi)\,d\xi}$$

with $\boxed{R(x,\xi) = \dfrac{v_1(x)v_2(\xi) - v_2(x)v_1(\xi)}{-pW}}$

notice $u(\alpha) = 0$

$u'(\alpha) = 0$.

Properties of R determine it uniquely:

a) R is antisymmetric: $R(x,\xi) = -R(\xi,x)$

b) $R(x,\xi)$ solves homogeneous DE (in x, also in ξ)

$$(pu')' + qu = 0$$

c) At $x = \xi$ it has initial values

$R(\xi,\xi) = 0$

$R_x(\xi,\xi) = \dfrac{1}{p(\xi)}$ check!

$R(x,\xi)$ is the influence function for f

also, 1-sided Green's fcn.

Notation $D(x,\xi) = \begin{vmatrix} v_1(x) & v_2(x) \\ v_1(\xi) & v_2(\xi) \end{vmatrix}$

$-pW = K$ (const)

So $R(x,\xi) = \dfrac{D(x,\xi)}{K}$

Example Solve

$$* \quad \begin{cases} u'' + u = f(x) & x > 0 \\ u(0) = 1 \\ u'(0) = -1 \end{cases}$$

Step 1 $u_H = c_1 \sin x + c_2 \cos x$.

$$u_p = \int_0^x R(x,\xi) f(\xi)\, d\xi \qquad \text{has } u_p(0) = u_p'(0) = 0$$

solves homogeneous DE $u_{xx} + u = 0$

$$u(x,\xi)\Big|_{x=\xi} = 0$$

$$u_x(\xi,\xi) = \frac{1}{p} = 1 \qquad (p(\xi) = 1).$$

$$\Rightarrow R(x,\xi) = \sin(x-\xi)$$

so soln to $*$ is

$$\boxed{u(x) = \int_0^x \sin(x-\xi) f(\xi)\, d\xi + \underbrace{\cos x - \sin x}_{u_H}}$$

you may have derived such sol'ns in 2280 via Laplace transform & Laplace transform convolution!

5.28

sign changed

BVP
$$\begin{cases} \frac{d}{dx}(pu') + qu = -f & \alpha < x < \beta \\ u(\alpha) = 0 \\ u(\beta) = 0 \end{cases}$$

$$u(x) = \int_\alpha^x R(x,\xi) f(\xi)\, d\xi + c_1 v_1 + c_2 v_2$$

$$u(\alpha) = 0 = c_1 v_1(\alpha) + c_2 v_2(\alpha)$$

$$u(\beta) = 0 = \underbrace{\int_\alpha^\beta -R(\beta,\xi) f(\xi)\, d\xi}_{-I} + c_1 v_1(\beta) + c_2 v_2(\beta).$$

$$\begin{bmatrix} v_1(\alpha) & v_2(\alpha) \\ v_1(\beta) & v_2(\beta) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{D} \begin{bmatrix} v_2(\beta) & -v_2(\alpha) \\ -v_1(\beta) & v_1(\alpha) \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} = \frac{I}{D} \begin{bmatrix} -v_2(\alpha) \\ v_1(\alpha) \end{bmatrix}$$

$D(\alpha,\beta)$
$\parallel$
discuss $D=0$:
if $v_1(\alpha)v_2(\beta) - v_1(\beta)v_2(\alpha) = 0$
then
$v_2(\beta)v_1(x) - v_1(\beta)v_2(x)$
solves
$$\begin{cases} (pu')' + qu = 0 \\ u(\alpha) = 0 \\ u(\beta) = 0. \end{cases}$$
So if the fully homogeneous problem has no non-zero soltns, $D \neq 0$.

So

$$R(x,\xi) \leftarrow \frac{D(x,\xi)}{K}$$

$$u(x) = \int_\alpha^x R(x,\xi) f(\xi)\, d\xi + \frac{-v_2(\alpha)}{D} v_1(x) \int_\alpha^\beta R(\beta,\xi) f(\xi)\, d\xi$$
$$+ \frac{v_1(\alpha)}{D} v_2(x) \int_\alpha^\beta R(\beta,\xi) f(\xi)\, d\xi$$

$$= \int_\alpha^\beta G(x,\xi) f(\xi)\, d\xi$$

$$K \cdot G(x,\xi) = \begin{cases} D(x,\xi) + \dfrac{D(\alpha,x)\, D(\beta,\xi)}{D(\alpha,\beta)} & \xi \leq x \\[2ex] \dfrac{D(\alpha,x)\, D(\beta,\xi)}{D(\alpha,\beta)} & \xi \geq x \end{cases}$$

because $D(x,x) = 0$ this fcn is continuous at $x = \xi$.

after algebra you get 28.2 p.121 which leads to symmetry, and (a)(b)(c)(d) 28.4 p 122

abcd $\longleftarrow$ from this formula symmetry is missing