Math 5440
Fri 11/12

overview and exam review notes

Exam Monday
JWB 240
any consecutive 80 minutes between 10:00–12:00 a.m.

exam to cover text 4.14 – 4.25 (we skipped 4.26)
5.27 – 5.29 (we didn't cover 5.30 yet)
as well as the supplementary topics we've covered in class.

Topics:

- separation of variables in PDE's
- how this leads to infinite sum solutions for the natural IBVP's, involving Fourier series Dirichlet / Neumann data, homogeneous heat & wave
  - homog. eqtns (IBVP's) in one space dimension
  - non-homogeneous PDE's, homogeneous data

superposition to solve problems with non-homogeneous data, non homogeneous PDE's.

- Fourier series
  inner product spaces ($\mathbb{R}$ & $\mathbb{C}$), length, distance, orthogonality, projection
  Fourier series in this general context
  special cases of sine series (zero Dirichlet data) & cosine series (zero Neumann data)
  convergence questions for Fourier series
    pointwise
    uniform
    in L^2 ("completeness")
- Using analysis, Fourier convergence theorems to show that formal separated solution series to homogeneous IVP's and IBVP's ~~actually~~ and BVP's actually solve the problems. Techniques include
  - interchanging differentiation and infinite sums
  - maximum principle
  - approximate identity arguments
- Using 1-variable Green's functions to solve some non-homogeneous PDE's with homogeneous boundary data, after separating variables from 2-dimensions

Some context and overview of what's to come next:
(and how it relates to what we've done so far)

- convolution with "point source" solutions (boundary source), to solve Dirichlet problems for homogeneous PDE's $\sim$ approximate identities!

```
> plot3d([r·cos(phi), r·sin(phi), Poissonkernel(r, phi)], r = 0..(.9), phi = 0..2·Pi, grid = [20, 40], scaling = constrained, orientation = [-110, 60], axes = boxed);
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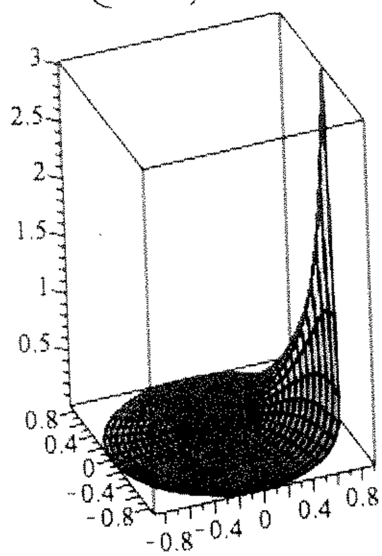
$$P(r\cos\varphi, r\sin\varphi) = \frac{1}{2\pi}\,\frac{1-r^2}{(1-r\cos\varphi)^2+(r\sin\varphi)^2}$$



soln to
$$\boxed{\begin{array}{l}\Delta u = 0 \text{ in unit disk}\\ u = f \text{ on unit circle}\end{array}}$$
is
$$u(r\cos\theta, r\sin\theta) = \int_{-\pi}^{\pi} f(\theta-\varphi)P(r,\varphi)\,d\varphi$$

```
> plot3d(Q(x, t, (0.2)), x = -2..2, t = 0..2, grid = [30, 30], axes = boxed, scaling = constrained);
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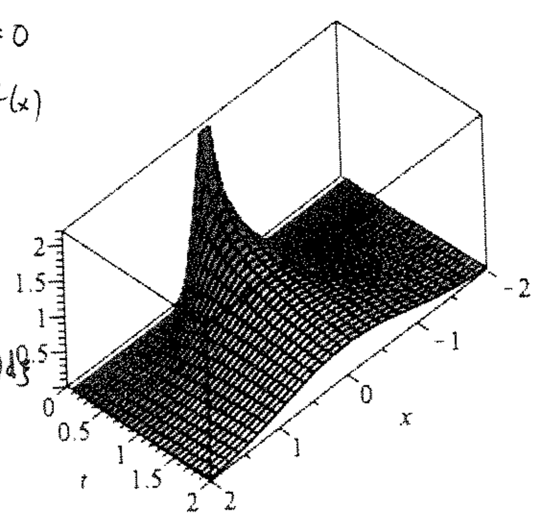
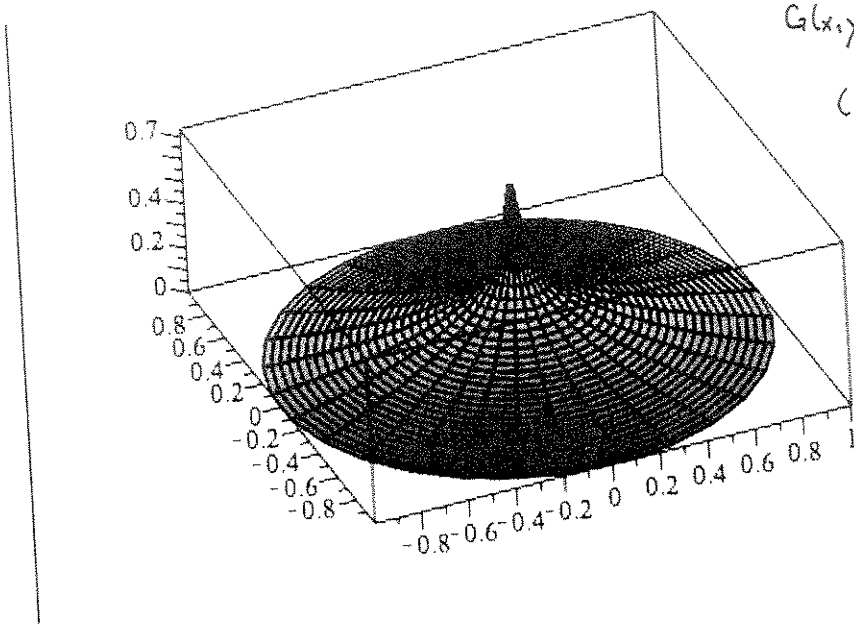
$$Q(x,t,k) = \frac{1}{2\sqrt{\pi k t}}\, e^{-\frac{1}{4}\frac{x^2}{kt}}$$ is point source for $u_t - ku_{xx} = 0$

soln to
$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = f(x) \end{cases}$$
is
$$u(x,t) = \int_{-\infty}^{\infty} f(x-\xi) \cdot Q(\xi,t,k)\,d\xi$$



- non-homogeneous problem $\Delta u = -F$
  (with & without homogeneous data on boundary)



$G(x,y) = -\frac{1}{2\pi} \ln r$

$(r = \sqrt{x^2+y^2})$

is a fundamental soltn for Laplace eqtn

$\Delta G = \delta(x,y)$, formally.

(In $\mathbb{R}^n$, $G(\vec{x}) = c_n \|\vec{x}\|^{2-n}$, $n>2$)

If F is a continuous function on $\mathbb{R}^2$ with compact support
zero outside a bounded set

then

$$u(\vec{x}) := \int_{\mathbb{R}^2} F(\vec{x}-\vec{\xi})\, G(\vec{\xi})\, d\xi$$

(x,y)

$$= \int_{\mathbb{R}^2} F(\xi)\, G(\vec{x}-\vec{\xi})\, d\xi$$

is a C^2 soltn to $\Delta u = F$

by adding a harmonic fcn to adjust the boundary data in any domain you can ~~sta~~ construct a Green's function (say for Dirichlet data) $G(\vec{x},\vec{\xi})$ s.t. soltn to

$$\begin{cases} \Delta u = -F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is $u(\vec{x}) = \int_{\Omega} G(\vec{x},\vec{\xi})\, F(\xi)\, d\xi$.

- $\Delta u = -F$ & natural BV's $\Rightarrow$ "self adjoint operator" on certain Hilbert function spaces $\Rightarrow$ complete orthonormal basis of eigenfcns for Δ. (~~Sq~~ Spectral thm)
  complete inner product spaces
  $\Rightarrow$ infinite series soltns for heat & wave eqtns, in which eigenfunctions represent fundamental modes.
  (generalization of sine, cos, & Fourier series soltn's)