Math 5440
Final exam
Tuesday Dec. 14.

Pick 5 of the 8 problems below. (Each is worth 30 points)
Good luck!

) What are the solutions to the IBVP's below?

(10 pts)

a) $\begin{cases} u_t - 4u_{xx} = 0 & 0<x<\pi,\ t>0 \\ u(x,0) = \sin x - \sin 3x & 0<x<\pi \\ u(0,t) = u(\pi,t) = 0 & t>0 \end{cases}$

(10 pts)

b) $\begin{cases} u_{tt} - 4u_{xx} = 0 & 0<x<\pi,\ t>0 \\ u(x,0) = \sin x & 0<x<\pi \\ u_t(x,0) = -\sin 3x & 0<x<\pi \\ u(0,t) = u(\pi,t) = 0 & t>0. \end{cases}$

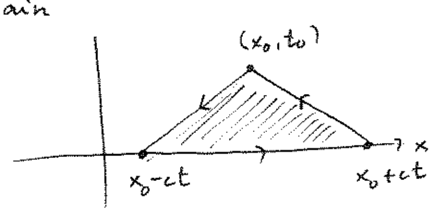
write your solution to (b) as a sum of product solutions

c) verify that your solution to (b) agrees with the d'Alembert solution. (10 pts)

) Consider the wave eqtn IVP (30 pts)

IVP $\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & x\in\mathbb{R},\ t\geq 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$

Apply Green's Theorem with the vector field $\langle M, N\rangle = \langle u_t, c^2 u_x\rangle$ and the triangle domain



to derive the solution formula for IVP

3)
Consider the heat equation initial boundary value problem

$$\text{IBVP}\begin{cases} u_t - k u_{xx} = 0 & 0<x<\pi,\ t>0 \\ u(x,0) = f(x) & 0<x<\pi \\ u(0,t) = u(\pi,t) = 0 & t>0 \end{cases}$$

a) Exhibit the (formal) series solution to IBVP above, obtained via separation of variables. You need not go through the separation of variables steps. (10)

b) Assuming $f(x)$ is the restriction of an ~~even~~ odd, 2π-periodic C^1 function to the interval $[0,\pi]$, with $f(\pi)=f(0)=0$, prove that the series in (a) converges uniformly to a function $u \in C([0,\pi]\times[0,\infty))$ which satisfies the initial and boundary values. Hints: you may quote the maximum principle and convergence facts for Fourier series (10)

c) Carefully explain why the series in (a) yields a function $u \in C^2((0,\pi)\times(0,\infty))$ which actually solves $u_t - k u_{xx} = 0$. (10)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain.

a) Prove the maximum principle for the Laplacian: If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $\Delta u \geq 0$ in Ω, then the maximum of u occurs on $\partial\Omega$.

Hint: first consider the case $\Delta u > 0$ in Ω, and apply the 2^{nd} derivative test then apply this case to $u_\varepsilon = u + \varepsilon |x|^2$, as $\varepsilon \to 0$. (20)

b) Use the maximum principle to prove that solutions $u \in C^2(\Omega) \cap C(\bar{\Omega})$ to the Dirichlet problem (10)

$$\begin{cases} \Delta u = F & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

are unique.

5) We derived the Green's function solution to

$$\text{BVP}\begin{cases}(pu')' + qu = -f & \alpha < x < \beta \\ u(\alpha) = u(\beta) = 0\end{cases}$$

arriving at a formula

$$u(x) = \int_\alpha^\beta f(\xi)\, G(x,\xi)\, d\xi$$

(providing the homogeneous DE had only the zero fcn sol'tn.

The Greens function $G(x,\xi)$ is characterized by

(i) for $x \neq \xi$ $G(x,\xi)$ solves homogeneous DE (in x)

(ii) $G(x,\xi)$ is cont. at $x = \xi$

(iii) $G(\alpha,\xi) = G(\beta,\xi) = 0$

(iv) $G_x\big|_{x=\xi^-} = G_x\big|_{x=\xi^+} + \frac{1}{p(x)}$.

a) Use these properties to find the (piecewise linear) $G(x,\xi)$ for the 1-d Laplace eqtn (20)

$$\begin{cases}u'' = -f & 0 < x < 1 \\ u(0) = u(1) = 0.\end{cases}$$

b) Show that $|G(x,\xi) - G(z,\xi)| \leq |x - z|$, i.e. G is Lipschitz cont in x with $L = 1$, for G in (a). (5)

c) Deduce that for $f \in L^2([0,1])$, the operator for inverse Laplace, (5)

$$Tf(x) = \int_0^1 f(\xi)\, G(x,\xi)\, d\xi$$

satisfies $|Tf(x) - Tf(z)| \leq \|f\|_{L^2}$, i.e. Tf is Lipschitz cont.

(this implies that the self-adjoint operator T is compact, because sequences of uniformly Lipschitz fcns have convergent subsequences in sup-norm - see Arzela-Ascoli theorem in analysis, and this implies L^2 convergence.)

6) Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with piecewise smooth boundary $\partial\Omega$.
Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solve

$$\begin{cases} \Delta u = F & \text{in } \Omega \\ u = h & \text{on } \partial\Omega \end{cases}$$

Let $G(x, x_0)$ be the Green's function for Δ.
Verify that for $x_0 \in \Omega$,

$$u(x_0) = \int_{\Omega} F(x) G(x, x_0) \, dV_n(x) + \int_{\partial\Omega} h(x) \frac{\partial G}{\partial n}(x, x_0) \, dV_{n-1}(x)$$

Hint: Recall

$$G(x, x_0) = K(x - x_0) + H(x, x_0)$$

where $K(x - x_0)$ is the fundamental (radial) soln for Δ,
and $H(x, x_0)$ is a harmonic function added to it
so that $G(x, x_0) = 0 \quad \forall x \in \partial\Omega$.

Apply an integral identity on $\Omega \setminus B_\varepsilon(x_0)$, and let $\varepsilon \to 0$.

7) We've studied the Navier Stokes' system of PDE's. It's

$$\begin{cases} \operatorname{div}(\rho \vec{u}) + \rho_t = 0 & \text{continuity eqtn} \quad \text{(conservation of mass)} \\ \rho \vec{u}_t + \rho \vec{u} \cdot \nabla \vec{u} = -\nabla p + \rho \vec{F} & \text{Navier-Stokes} \quad \text{(change in momentum)} \end{cases}$$

$\rho(\vec{x}, t) =$ density
$p(\vec{x}, t) =$ pressure
$\vec{F}(\vec{x}, t) =$ body force/mass
$\vec{u}(\vec{x}, t) =$ velocity field of particle motion

a) The continuity equation is the easier of the two to derive – derive it

b) We linearized these equations under the further assumption that pressure was a function of density. We arrived at

$$\begin{cases} \rho_0 \operatorname{div} \vec{u} + \rho_t = 0 \\ \rho_0 \vec{u}_t + p'(\rho_0) \nabla \rho = \rho_0 \vec{F} \end{cases}$$

Use this system to show that in this linearized model the density $\rho(\vec{x}, t)$ satisfies the 3 space dimension wave eqtn

$$\rho_{tt} - c^2 \Delta \rho = -\rho_0 \operatorname{div} \vec{F} \qquad c^2 = p'(\rho_0)$$

8) a) Let $f(\theta)$ be a 2π-periodic continuous function.
Exhibit the formal solution to Laplace's equation in the unit disk

$$\begin{cases} \Delta u = 0 & \text{in } D_1(0) \subset \mathbb{R}^2 \\ u = f & \text{on } \partial D_1 = S^1 \end{cases} \qquad (10)$$

in terms of polar coordinates, $u(x,y) = \tilde{u}(r,\theta)$.

b) Explain how to rewrite the solution in part (a), for $r<1$, as

$$\tilde{u}(r,\theta) = \int_{-\pi}^{\pi} \frac{1}{2\pi}\left(1 + 2\sum_{n=1}^{\infty} r^n \cos n(\theta-\varphi)\right) f(\varphi)\,d\varphi \qquad r<1$$

Hint: write out Fourier coeff's, justify interchanging summation & integration
use trig id. (10)

c) Through Euler formula magic we showed that the "Poisson kernel"

$$P_r(\varphi) := \frac{1}{2\pi}\left(1 + 2\sum_{n=1}^{\infty} r^n \cos n\varphi\right) = \frac{1}{2\pi}\frac{1-r^2}{(1-r\cos\varphi)^2 + (r\sin\varphi)^2}$$

So (using our usual convolution COV),

$$\tilde{u}(r,\theta) = \int_{-\pi}^{\pi} f(\theta-\varphi)\,P_r(\varphi)\,d\varphi.$$

$P_r(\varphi)$ has <u>approximate identity</u> properties as $r \nearrow 1$.
Use these properties to prove that if we define

$$\tilde{u}(1,\theta) := f(\theta)$$

then our separated soltn to (a) actually extends continuously to the boundary (even though the Fourier series for f might not have even converged pointwise.
(10)