Math 5440
Exam 2 Solns

1a) $u_t - k u_{xx} = 0 \quad 0<x<\pi \quad t>0$
$u(x,0) = f(x) \quad 0<x<\pi$
$u_x(0,t) = u_x(\pi,t) = 0 \quad t>0$

$U = X(x)T(t) \quad HE \Rightarrow \quad T'X - kX''T = 0$
$T'X = kX''T$
$\frac{T'}{kT} = \frac{X''}{X} = \lambda$ const

$X'' - \lambda X = 0$
$X'(0) = X'(\pi) = 0.$

Case I $\lambda = 0$: $X(x) = ax+b$
BC $\Rightarrow a = 0$
$X = \text{span}\{1\}$
$T' = 0 \Rightarrow$
$\boxed{XT \in \text{span}\{1\}}$

Case II
$\lambda = \omega^2 > 0$
$X = \text{span}\{e^{\omega x}, e^{-\omega x}\}.$
$X(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$
$X'(x) = c_1\omega e^{\omega x} - c_2\omega e^{-\omega x}$
$X'(0) = 0 = [1 \ \ -1]\begin{bmatrix} c_1\omega \\ c_2\omega \end{bmatrix}$

$X'(\pi) = 0 = [e^{\omega\pi} \ \ -e^{-\omega\pi}]\begin{bmatrix} c_1\omega \\ c_2\omega \end{bmatrix}$

$\begin{bmatrix} X'(0) \\ X'(\pi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ e^{\omega\pi} & -e^{-\omega\pi} \end{bmatrix}\begin{bmatrix} c_1\omega \\ c_2\omega \end{bmatrix} \Rightarrow c_1 = c_2 = 0$ no non-zero solns

$\det = e^{\omega\pi} - e^{-\pi\omega}$
$= 2\sinh\omega\pi \neq 0$

Case III
$\lambda = -\omega^2 < 0$
$X'' + \omega^2 X = 0$

$X = c_1\cos\omega x + c_2\sin\omega x$
$X'(x) = -c_1\omega\sin\omega x + c_2\omega\cos\omega x$
$X'(0) = 0 = c_2\omega \Rightarrow c_2 = 0$
$X'(\pi) = 0 \Leftrightarrow \sin\omega\pi$ iff $\omega = n,\ n\in\mathbb{Z}.$
since $\sin nx = -\sin(-nx)$ restrict to $n \in \mathbb{N}$.

$\Rightarrow \omega = n,\ \lambda = -n^2,\ n\in\mathbb{N},\ X(x) = \text{span}\{\cos nx\}$

$\frac{T'(t)}{kT} = -n^2 \qquad T' = -kn^2 T \qquad T = T_0 e^{-kn^2 t}$

$\Rightarrow \boxed{X(x)T(t) = \text{span}\left\{e^{-kn^2 t}\cos nx\right\}_{n\in\mathbb{N}}}$

1b) For Neumann BC extend f even

$\Rightarrow f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$$\boxed{u(x,t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\, e^{-kn^2 t}}$$

1c) $f \in C^1 \Rightarrow f'$ cont.

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \Rightarrow f' \sim \sum_{n=1}^{\infty} -n a_n \sin nx$$

Parseval for $f' \Rightarrow \sum_{n=1}^{\infty} n^2 a_n^2 < \infty$

$f \in C^1 \Rightarrow$ Fourier series converges uniformly to f.

And, so,

$$|f(x) - S_N(f;x)| = \left| \sum_{n=N+1}^{\infty} a_n \cos nx \right|$$

$$\leq \sum_{N+1}^{\infty} |a_n| \quad ; \quad |a_n| = (n|a_n|)\frac{1}{n}$$

$$\leq \left(\sum_{N+1}^{\infty} |a_n|^2 n^2\right)^{1/2} \left(\sum_{N+1}^{\infty} \frac{1}{n^2}\right)^{1/2}$$

$$\downarrow N \to \infty.$$

So $S_N(f) \to f$ with this uniform estimate 0

alternately you can use the Maximum principle: by the strong max principle the max of $|u_N - u_M|$ must occur at points $(x,0)$ so $|u_N - u_M|_{max} \leq |S_N(f) - S_M(f)| \to 0$

By the estimate, for $N > M$

sum up to $n=N$

$$|u_N(x,t) - u_M(x,t)| = \left| \sum_{n=M+1}^{N} a_n \cos nx \, e^{-kn^2 t} \right| \leq \sum_{M+1}^{N} |a_n| \longrightarrow 0 \quad \text{as } M < N, \; M,N \to \infty.$$

thus $\{u_N\}$ is a uniformly Cauchy sequence of continuous fcns $[0,\pi] \times [0,\infty)$

$\Rightarrow$ converges uniformly to

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \, e^{-kn^2 t}, \quad \text{with } u(x,0) = f(x).$$

for $t \geq \delta > 0$ the series $\sum a_n \cos nx \, e^{-kn^2 t}$ and any finite iteration of x & t derivative series converge ~~abs~~ uniformly absolutely (since $\sum |a_n| n^{\ell} e^{-kn^2 \delta}$ converges and $\ell \in \mathbb{N}$)

so $u(x,t)$ is C^∞ for $t > 0$ and the derivatives of u are just the series of derivatives.

in particular

- $u_t - k u_{xx} = 0$, because this is true for each term.
- $u_t(0,t) = u_t(\pi,t) = 0 \;\; \forall \, t > 0$ because true for each term.

2) $u_{tt} - 4u_{xx} = 0 \quad 0 < x < \pi \quad t > 0$

$u(x,0) = \sin^3 x \quad 0 < x < \pi$

$u_t(x,0) = 0 \quad$ "

$u(0,t) = u(\pi, t) = 0 \quad t > 0.$

2a) $u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct))$

$$\boxed{u(x,t) = \frac{1}{2}\left(\sin^3(x-2t) + \sin^3(x+2t)\right)}$$

2b) $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

if $u = X(x)T(t)$

$XT'' - 4X''T = 0$

$XT'' = 4X''T$

$\frac{T''}{4T} = \frac{X''}{X}$

$X(x) = \cancel{\{\sin nx\}}$ span $\{\sin nx\}_{n \in \mathbb{N}}$

$\Rightarrow \frac{X''}{X} = -n^2$

$\Rightarrow \frac{T''}{T} = -4n^2$

$\Rightarrow T(t) = \text{span}\{\cos 2nt, \sin nt\}$

for $X(x) = \sin nx \quad n \in \mathbb{N}$

and for $u_t(x,0) = 0$

we want $XT = (\sin nx)(\cos 2nt)$

thus our separated variables sol'n is

$$\boxed{u(x,t) = \frac{3}{4}\sin x \cos 2t - \frac{1}{4}\sin 3x \cos 6t}$$

to show $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

one way is via Euler:

$$(e^{ix})^3 = e^{3ix}$$

$$(\cos x + i\sin x)^3 = \cos 3x + i\sin 3x$$

$$\cos^3 x + 3i\cos^2 x \sin x - 3\cos x \sin^2 x - i\sin^3 x = \cos 3x + i\sin 3x$$

take imag parts $\Rightarrow \sin 3x = 3\cos^2 x \sin x - \sin^3 x = 3(1-\sin^2 x)\sin x - \sin^3 x$

$= 3\sin x - 4\sin^3 x$

$4\sin^3 x = 3\sin x - \sin 3x$

$$\boxed{\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x}$$

2c) 2a) soln is $\frac{1}{2}\left(\sin^3(x-2t) + \sin^3(x+2t)\right)$

$$= \frac{1}{2}\left[\frac{3}{4}\sin(x-2t) - \frac{1}{4}\sin 3(x-2t) + \frac{3}{4}\sin(x+2t) - \frac{1}{4}\sin 3(x+2t)\right]$$

$\frac{3}{4}\left(\sin x \cos 2t - \cancel{\cos x \sin 2t}\right)$

$-\frac{1}{4}\left(\sin 3x \cos 6t - \cancel{\cos 3x \sin 6t}\right)$

$\frac{3}{4}\left(\sin x \cos 2t + \cancel{\cos x \sin 2t}\right)$

$-\frac{1}{4}\left(\sin 3x \cos 6t + \cancel{\cos 3x \sin 6t}\right)$

$$= \frac{1}{2}\left[\frac{3}{2}\sin x \cos 2t - \frac{1}{2}\sin 3x \cos 6t\right]$$

$$= \frac{3}{4}\sin x \cos 2t - \frac{1}{4}\sin 3x \cos 6t$$

✓

3. $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

3a) $U = R(r)\Theta(\theta)$

$LU = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$

$\frac{r^2}{R\Theta} \cdot \left(R''\Theta + \frac{1}{r}R'\Theta = -\frac{1}{r^2}R\Theta'' \right)$

$\frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$ Θ 2π-periodic

$\Theta'' - \lambda\Theta = 0$

Case I

$\lambda = 0 \Rightarrow \Theta = \text{span}\{1\}$

$r^2R'' + rR' = 0$

see below.

Case II $\lambda = \omega^2$

$\Theta'' - \omega^2\Theta = 0$

$\Theta = \text{span}\{\cosh\omega\theta, \sinh\omega\theta\}$

no linear combo of $\cosh\omega\theta$, $\sinh\omega\theta$ is periodic.

no solts.

Case III $\lambda = -\omega^2$

$\Theta'' + \omega^2\Theta = 0$

$\Rightarrow \Theta(\theta) = \text{span}\{\cos n\theta, \sin \theta\}_{n\in\mathbb{N}}$

$\lambda = -n^2$

$R(r)$:

$\frac{r^2R''}{R} + \frac{rR'}{R} = n^2$

$* \quad r^2R'' + rR' - n^2R = 0$

$r = e^t$

$\frac{dR}{dt} = \frac{dR}{dr}e^t$; $\frac{d^2R}{dt^2} = \frac{d^2R}{dr^2}e^{2t} + \frac{dR}{dr}e^t = R''(r)r^2 + R'(r)r$

so $*$ is $\frac{d^2R}{dt^2} - n^2R(t) = 0$

$n = 0 \Rightarrow \frac{d^2R}{dt^2} = 0 \quad R = c_1 + c_2 t$

$R(r) = c_1 + c_2 \ln r.$

$\boxed{n=0: \text{span}\{1, \ln r\}}$

$n \geq 1$: $R(t) = \text{span}\{e^{nt}, e^{-nt}\}$

$R(r) = \text{span}\{r^n, r^{-n}\}$.

$\boxed{n\in\mathbb{N}: \text{span}\{r^n\cos n\theta, r^n\sin n\theta, r^{-n}\cos n\theta, r^{-n}\sin n\theta\}.}$

3b) $u(1,\theta) = \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

$\boxed{\text{so } u(r,\theta) = \frac{3}{4}r\sin\theta - \frac{1}{4}r^3\sin 3\theta}$

3c) $u(r,\theta)$ is really $\bar{u}(r\cos\theta, r\sin\theta)$ where $x = r\cos\theta$, $y = r\sin\theta$

Since $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$r^3\sin 3\theta = 3r^3\sin\theta - 4(r\sin\theta)^3$

$= 3(x^2+y^2)y - 4y^3$

$= 3x^2y - y^3$

so, $\boxed{\bar{u}(x,y) = \frac{3}{4}y - \frac{1}{4}(3x^2y - y^3)}$

(notice $\bar{u}_{xx} + \bar{u}_{yy}$

$= -\frac{1}{4}6y + \frac{1}{4}6y = 0$ ✓)

4. a) $(pu')' + qu = -f$

$$u(x) = \int_\alpha^\beta f(\xi)G(x,\xi)d\xi = \int_\alpha^x f(\xi)G(x,\xi)d\xi + \int_x^\beta f(\xi)G(x,\xi)d\xi$$

so $u' = \frac{d}{dx}(\%) = f(x)\cancel{G(x,x)} + \int_\alpha^x f(\xi)G_x(x,\xi)d\xi - f(x)\cancel{G(x,x)} + \int_x^\beta f(\xi)G_x(x,\xi)d\xi$

(using diff rules)

so $pu' = \int_\alpha^x f(\xi)\, \cancel{p(x)}\, p(x)G_x(x,\xi)\,d\xi + \int_x^\beta f(\xi)\,p(x)G_x(x,\xi)d\xi$

($p(x)$ is const wrt ξ, so moved it into integral)

so $(pu')' = f(x)\,pG_x\Big|_{\xi=x^-} + \int_\alpha^x f(\xi)(pG_x)_x\,d\xi + \int_x^\beta f(\xi)(pG_x)_x\,d\xi - f(x)\,pG_x\Big|_{\xi=x^+}$

$= f(x)\left(pG_x\Big]_{\xi>x}^{\xi<x} \right) + \int_\alpha^\beta f(\xi)\left[\cancel{G} - q\cancel{u}\,G(x,\xi)\right]$

because $(pG_x)_x = -qG$ for $\xi<x$ and for $\xi>x$ (by (i)).

$= -f(x)$

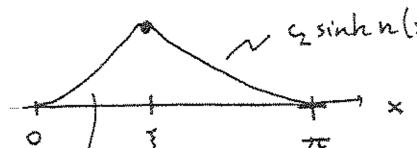
(by (iv))

so $(pu')' = -f - qu$ ■

notice $u(\alpha) = u(\beta) = 0$ since $G(\alpha,\xi) = G(\beta,\xi) = 0$ $\forall \xi$. ■

4b) $\begin{cases} u''(x) - n^2u(x) = -f \\ u(0) = u(\pi) = 0 \end{cases}$

soltns to $u'' - n^2u = 0$ are $\text{span}\{e^{nx}, e^{-nx}\} = \text{span}\{\sinh nx, \cosh nx\}$.



$G = c_2 \sinh n(x-\pi)$

$G_x = c_2 n \cosh n(x-\pi)$

$G = c_1 \sinh nx$

$G_x = c_1 n \cosh nx$

$\begin{cases} c_1 \sinh n\xi = c_2 \sinh n(\xi-\pi) \\ c_1 n \cosh n\xi = c_2 n \cosh n(\xi-\pi) + 1 \end{cases}$ cont @ $\xi = x$

$$\begin{bmatrix} \sinh n\xi & -\sinh n(\xi-\pi) \\ n\cosh n\xi & -n\cosh n(\xi-\pi) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

det $= -n\left[\sinh n\xi \cosh n(\xi-\pi) - \cosh n\xi \sinh n(\xi-\pi)\right]$

$+ n\left[\sinh(n(\xi-\pi) - n\xi)\right]$

$= -n\sinh n\pi$

$$\boxed{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{n\sinh n\pi}\begin{bmatrix} * & \sinh n(\xi-\pi) \\ * & \sinh n\xi \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{n\sinh n\pi}\begin{bmatrix} \sinh n(\xi-\pi) \\ \sinh n\xi \end{bmatrix}}$$