Exam 1 solns

1) $\partial_t^2 - c^2\partial_x^2 = (\partial_t - c\partial_x)(\partial_t + c\partial_x)$

Let $\dfrac{\partial}{\partial \xi} = \dfrac{\partial}{\partial t} - c\dfrac{\partial}{\partial x}$ $\qquad$ since $u_\xi = u_t t_\xi + u_x x_\xi$

$\dfrac{\partial}{\partial \eta} = \dfrac{\partial}{\partial t} + c\dfrac{\partial}{\partial x}$ $\qquad$ $u_\eta = u_t t_\eta + u_x x_\eta$

deduce $t_\xi = 1 \quad x_\xi = -c$

$t_\eta = 1 \quad x_\eta = c$

$\Rightarrow t = \xi + \eta$

$x = c(-\xi + \eta)$ $\qquad$ $\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -c & c \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}$

$u_{tt} - c^2 u_{xx} = 0$ $\qquad$ $\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{1}{2c}\begin{bmatrix} c & -1 \\ c & 1 \end{bmatrix}\begin{bmatrix} t \\ x \end{bmatrix}$

iff $\bar{u}_{\xi\eta} = 0$

iff $\bar{u}_\xi = \int \bar{u}_{\xi\eta}\, d\eta = \tilde{p}(\xi)$

$\Leftrightarrow \bar{u} = \int \bar{u}_\xi\, d\xi = \int \tilde{p}(\xi)\, d\xi + q(\eta) = p(\xi) + q(\eta)$

$\Leftrightarrow u = p\left(\frac{1}{2c}(ct - x)\right) + q\left(\frac{1}{2c}(ct + x)\right)$

$= Q(x - ct) + P(x + ct)$ ∎

2) $\begin{cases} u_{tt} - 4u_{xx} = 1 & 0 < x < 2,\ t > 0 \\ u(x,0) = 0 & \\ u_t(x,0) = \sin\frac{\pi}{2}x & 0 < x < 2 \\ u(0,t) = u(2,t) = 0 & \end{cases}$

for odd extension of data to $\mathbb{R}$ (or $\mathbb{R}\times\mathbb{R}^+$ for F)

i.e. odd at $x=0$ and $x=2$,

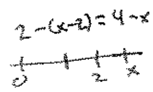
we can use soltn formula

$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right)$

$+ \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x} + \frac{1}{2c}\iint F\, dA$

in formula at right,

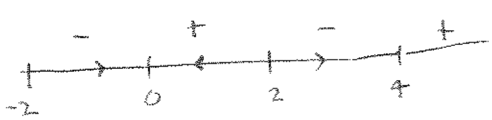
$f \equiv 0$, so 1^{st} term $= 0$.



$g(x) = \sin\frac{\pi}{2}x$ is odd wrt $x = 0$ and $x = 2$ $\quad (g(x) = -g(4-x))$

extend $F(x,t) = 1$ s.t. $F = -1$ for $x \in (-2,0)$

$F = -1$ for $x \in (2,4)$



etc.

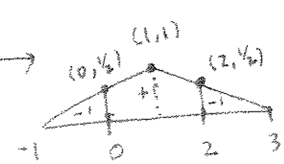
$c = 2$.

$u(1,1) = \frac{1}{4}\int_{1-2}^{1+2} \sin\left(\frac{\pi}{2}\bar{x}\right)d\bar{x} + \frac{1}{4}\iint \pm 1\, dA = 0 + \frac{1}{4}(1) = \boxed{\frac{1}{4}}$

$\frac{1}{4}\left(-\frac{2}{\pi}\cos\frac{\pi}{2}\bar{x}\right)\Big|_{-1}^{3}$

0

$\longrightarrow$



$\iint F = -1\left(\frac{1}{4}\right) + 1\left(\frac{3}{4}\right) + 1\left(\frac{3}{4}\right) - \frac{1}{4}$

$= 1$

3) as Michael brought up in exam, Neumann problem soltns are only unique up to additive constant.

If w is any fcn, $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$,

$$\int_{\Omega} \operatorname{div}(w \nabla w) = \int_{\partial\Omega} w \nabla w \cdot n$$

$$\int_{\Omega} w \Delta w + |\nabla w|^2 = \int_{\partial\Omega} w \nabla w \cdot n$$

If $w = u - v$ where u, v are two soltns to either DP or NP, then the boundary integral on right $= 0$, since $w = 0$ on $\partial\Omega$ for DP
$\nabla w \cdot n = 0$ on $\partial\Omega$ for NP.

Also $\Delta w = 0$ in both cases.

Deduce $\int_{\Omega} |\nabla w|^2 = 0 \Rightarrow |\nabla w| \equiv 0 \Rightarrow w = \text{const.}$

for DP, $w = 0$ on $\partial\Omega \Rightarrow w \equiv 0$ ∎

for NP just get $w = c$; $u = v + c$.

4) a) If $w_t - k\Delta w < 0$ in $\Omega \times (0, T]$

since $w \in C(\bar{\Omega} \times [0,T])$ it attains its maximum value, call this value M.

We show this cannot happen at any $(x,t) \in \Omega \times (0,T]$
which means it can only happen at
$(x,t) \in (\partial\Omega \times [0,T]) \cup (\Omega \times \{0\})$.

<u>proof</u>: Let $u(x_0, t_0) = M$
$(x_0, t_0) \in \Omega \times (0,T]$

$\Rightarrow \nabla u(x_0, t_0) = \vec{0}$

$\boxed{u_t(x_0, t_0) \geq 0}$ $\quad (u_t(x_0, t_0) = 0$ if $t_0 < T)$.

Since $\nabla u(x_0, t_0) = \vec{0}$ and $u(x_0, t_0)$ is local max
$u_{\xi\xi}(x_0, t_0) \leq 0$ $\forall$ spatial 2^{nd} derivatives

$\boxed{\Rightarrow \Delta u(x_0, t_0) \leq 0}$

$\Rightarrow u_t - k\Delta u \geq 0$ at (x_0, t_0) ⨳. ∎

b) Let u, v be 2 soltns, $w = u - v$.
Then w satisfies

$$w_t - k\Delta w = 0 \quad \text{in } \Omega \times (0, \infty)$$
$$w(x,t) = 0 \quad \text{on } \partial\Omega \times [0, \infty)$$
$$w(x,0) = 0 \quad \text{on } \bar{\Omega}$$

Max principle $\Rightarrow \boxed{w \leq 0}$ on $\bar{\Omega} \times [0, \infty)$
but $(-w)$ satisfies same homogeneous data,
so $(-w) \leq 0$ on $\bar{\Omega} \times [0, \infty)$
i.e. $\boxed{w \geq 0}$

$\Rightarrow w \equiv 0 \Rightarrow u \equiv v$ ∎

5a). let Ω be a test domain. (fixed)

conservation of mass $\Rightarrow$ $\frac{d}{dt}\left(\int_{\Omega} \rho \, dV\right) = \int_{\partial\Omega} (\rho\vec{u})\cdot\vec{n}\, dS$ (flux thru $\partial\Omega$)

$$\int_{\Omega} \rho_t \, dV = \int_{\partial\Omega} \rho\vec{u}\cdot\vec{n}\, dS$$

$$= \int_{\Omega} \operatorname{div}(\rho\vec{u})\, dV$$

i.e $\int_{\Omega} (\rho_t - \operatorname{div}(\rho\vec{u}))\, dV = 0$

for all test domains Ω

assuming the integrand is a continuous fn this implies $\boxed{\rho_t - \operatorname{div}(\rho\vec{u}) = 0}$

5b) $\rho_0 \operatorname{div}\vec{u} + \rho_t = 0$ (1)

$\rho_0 \vec{u}_t + p'(\rho_0)\nabla\rho = \rho_0 \vec{F}$ (2)

$\partial_t(1) - \operatorname{div}(2) \Rightarrow \cancel{\rho_0 \operatorname{div}\vec{u}_t} + \rho_{tt} - \cancel{\rho_0 \operatorname{div}(\vec{u}_t)} - p'(\rho_0)\operatorname{div}\nabla\rho = -\rho_0 \operatorname{div}\vec{F}$

$$\boxed{\rho_{tt} - p'(\rho_0)\Delta\rho = -\rho_0 \operatorname{div}\vec{F}}$$