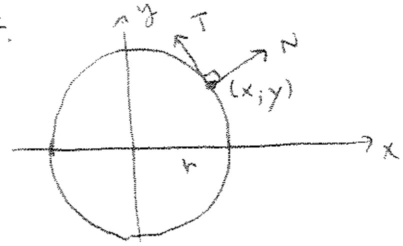
Math 5440
Mon Dec. 6.

a clean way to separate variables with radial & spherical directions.

Recall Δ operator is invariant wrt rotated coord systems.

$\underline{n=2}$.



let T, N be unit tang, normal to circle at (x,y)

$$\Delta u = u_{xx} + u_{yy} = u_{TT} + u_{NN}.$$

write

$$u(x,y) = u(r\cos\theta, r\sin\theta) = \bar{u}(r,\theta).$$

$$\vec{\varphi}(t) = \begin{bmatrix} r\cos\frac{t}{r} \\ r\sin\frac{t}{r} \end{bmatrix} \quad \text{has} \quad \varphi'(t) = \begin{bmatrix} -\sin t/r \\ \cos t/r \end{bmatrix} = T$$

let $f(t) = u(\vec{\varphi}(t)) = \bar{u}(r, t/r)$

$$f'(t) = \nabla u(\vec{\varphi}(t)) \cdot \varphi'(t) = \frac{1}{r}\bar{u}_\theta(r, t/r)$$

$u_x x' + u_y y'$

$$f''(t) = \underbrace{T^T \begin{bmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{bmatrix} \vec{T}}_{u_{TT}} + \nabla u \cdot \vec{\varphi}''(t) = \frac{1}{r^2}\bar{u}_{\theta\theta}(r, t/r)$$

$-\frac{1}{r}\begin{bmatrix} \cos t/r \\ \sin t/r \end{bmatrix}$ $\qquad \bar{u}_{\theta\theta} = \Delta_{S^1}\bar{u}(r,\theta)$.

Laplacian on unit circle

so $\Delta u = u_{NN} + u_{TT} = \bar{u}_{rr} + \frac{1}{r}\bar{u}_r + \frac{1}{r^2}\bar{u}_{\theta\theta}$

$n=3$

$$\Delta u = u_{TT} + u_{SS} + u_{NN} \qquad u(x,y,z) = \bar{u}(r,\vec{\omega}) \qquad \vec{\omega} = \frac{\vec{x}}{\|\vec{x}\|} \in S^2_1$$

parameterize $\perp$ great circle geodesics by arclength, at point $\omega \in S^2$
repeat computation above. The sum of the two second derivatives of u along the great circles is called $\Delta_{S^2}\bar{u}$.

Deduce $\boxed{\Delta u = \bar{u}_{rr} + \frac{2}{r}\bar{u}_r + \frac{1}{r^2}\Delta_{S^2}\bar{u}}$

(generalizes to $\mathbb{R}^n$: $\quad u(\vec{x}) = \bar{u}(r,\omega) \qquad r = |x|$

$\omega = \frac{x}{|x|}$.

$$\boxed{\Delta u = \bar{u}_{rr} + \frac{n-1}{r}\bar{u}_r + \frac{1}{r^2}\Delta_{S^2}\bar{u}}$$

HE $\quad u_t - k\Delta u = 0$

WE $\quad u_{tt} - c^2\Delta u = 0$

when you separate variables in ~~$\mathbb{R}^2$~~ unit disk for wave eqtn you get $u = RT\Theta$

$$T'' + c^2\lambda T = 0$$

$$\Theta'' + \gamma\Theta = 0$$

$$R'' + \frac{1}{r}R' + \left(\lambda - \frac{\gamma}{r^2}\right)R = 0$$

$$\Rightarrow \Theta_n(\theta) = \begin{cases} \text{span}\{1\} & n = 0 \\ \text{span}\{\cos n\theta, \sin n\theta\} & n \in \mathbb{N}. \end{cases}$$

so $R(r)$ sats

$$\begin{cases} R'' + \frac{1}{r}R' + \left(\lambda - \frac{n^2}{r^2}\right)R = 0 \\ R(0) \text{ finite} \\ R(1) = 0 \end{cases}$$

$\lambda = 0$ no soltn except 0

so $\lambda > 0$

$\rho = \sqrt{\lambda}\, r$ scale $\qquad \frac{1}{\sqrt{\lambda}}\rho = r$

$R_r = R_\rho \rho_r = \sqrt{\lambda} R_\rho$

$R_{rr} = \lambda R_{\rho\rho}$

yields

$$\boxed{R_{\rho\rho} + \frac{1}{\rho}R_\rho + \left(1 - \frac{n^2}{\rho^2}\right)R = 0}$$

Bessel's eqtn of order n.
solve with power series

get $J_1(\rho), J_2(\rho), J_3(\rho), \ldots J_n(\rho)$, each with ∞ zeroes.

Scale back to satisfy $R(1) = 0$, this dets λ's.

if $J_n(\rho_k) = 0$ then

$J_n(r\rho_k)$ has a zero at $r = 1$, eigenvalue $\sqrt{\rho_k}$

3-d WE

$$T'' + c^2 \lambda T = 0$$

$$\Delta_{S^2} \Omega + \gamma \Omega = 0$$

$$R_{rr} + \frac{2}{r} R_r + \left(\lambda - \frac{\gamma}{r^2}\right) R = 0$$

analogous transformation for $R(r)$ (to different Bessel fcns), once we know the γ's.

Here's the miracle (in all $\mathbb{R}^n$).

See spherical harmonics page. "Higher dimensions"

P_ℓ = homogeneous polys of degree ℓ

e.g. in $\mathbb{R}^2$, $P_1 = \text{span}\{x, y\}$

$P_2 = \text{span}\{x^2, xy, y^2\}$

etc.

$A_\ell \subset P_\ell$. A_ℓ = harmonic polys of degree ℓ.

$A_1 = P_1$

$A_2 = \text{span}\{xy, x^2 - y^2\}$

etc.

H_ℓ = restrict A_ℓ to S^{n-1}. $H_1 = \text{span}\{\cos\theta, \sin\theta\}$

$H_2 = \text{span}\{\cos 2\theta, \sin 2\theta\}$

uses Stone-Weierstrass Theorem

Theorem $\text{span}\{H_\ell\}_{\ell = 0,1,2,\ldots}$ is dense in $C(S^{n-1})$ in uniform (sup norm), so also in L^2

$$f \in H_\ell \Rightarrow \boxed{\Delta_{S^{n-1}} f = -\ell(\ell + n - 2) f}$$

e.g. in $\mathbb{R}^2$, $f_{\theta\theta} = -\ell^2 f$ ✓

in $\mathbb{R}^3$, $\boxed{\Delta_{S^2} f = -\ell(\ell+1) f}$

so these are eigenfcns of spherical Laplacian.

by density then they provide (after $\perp$ & $\|\cdot\| = 1$) a complete o.n. basis

We can actually check the eigenvalues

$$f \in H_\ell$$

$$\Rightarrow \quad r^\ell f \in A_\ell$$

$$\Rightarrow \quad \Delta(r^\ell f) = 0 = (r^\ell f(\omega))_{rr} + \frac{n-1}{r}(r^\ell f(\omega))_r + \frac{1}{r^2}\Delta_{S^{n-1}}(r^\ell f(\omega))$$

$$= r^{\ell-2}\Big[\big(\underbrace{\ell(\ell-1) + (n-1)\ell}_{\substack{\ell(\ell-1+n-1)\\ =\ell(\ell+n-2)}}\big) f + \Delta_{S^{n-1}} f\Big]$$