Friday Dec 3
Math 5440

continuing spectral theorem discussion...
(we're on page 9...)

example 2 cont'd:

so for $\Omega \subset \mathbb{R}^n$ bounded we have an orthonormal, complete, collection of eigenfunctions for the Δ, $\{f_k\}$

$$\begin{cases} \Delta f_k = -\lambda_k f_k & \text{in } \Omega \\ f_k = 0 & \text{on } \partial\Omega \end{cases}$$

$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \quad \lambda_k \to \infty.$

$\uparrow$
it turns out $\lambda_1 < \lambda_2$ is strict, and $\dim(E_{\lambda_1}) = 1$, i.e. there is one eigenfunction,
called the principle eigenfunction (up to scaling)
corresponds to the lowest vibration mode in wave eqtn: $k=1$, for sep. soltns

$$u_k(x,t) = f_k(x) \cdot \text{span}(\cos \omega_k ct, \sin \omega_k ct)$$

$$\omega_k = \sqrt{\lambda_k}$$

There is a variational way to find the eigenspaces and eigenvalues, called Rayleigh-Ritz method (also in example 3, see §7.36 in text).

It turns out that if $u \in W^{1,2}(\Omega)$ with $u = 0$ on $\partial\Omega$ $\to$ space called $W_0^{1,2}(\Omega)$
(think of piecewise C^1 function)

then both $\quad u = \sum_{k=1}^{\infty} c_k f_k \quad$ and $\quad \nabla u = \sum_{k=1}^{\infty} c_k \nabla f_k \quad$ converge in $L^2(\Omega)$.

Thus $\quad \|u\|^2 = \int_\Omega |u|^2 = \lim_{N\to\infty} \left\| \sum_{k=1}^{N} c_k f_k \right\|^2 = \sum_{k=1}^{\infty} c_k^2$

and $\quad \|\nabla u\|^2 = \lim_{N\to\infty} \left\| \sum_{k=1}^{N} c_k \nabla f_k \right\|^2 = \lim_{N\to\infty} \sum_{k=1}^{N} \lambda_k c_k^2 = \sum_{k=1}^{\infty} \lambda_k c_k^2$

$$\langle \nabla f_k, \nabla f_\ell \rangle = \int_\Omega \nabla f_k \cdot \nabla f_\ell = -\int_\Omega f_k \underbrace{\Delta f_\ell}_{\lambda_\ell f_\ell} = \begin{cases} 0 & k \neq \ell \\ \lambda_k & k = \ell \end{cases}$$

Thus $\lambda_1 = \min_{u \in W_0^{1,2}(\Omega)} \dfrac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} = \min_{\substack{u \in W_0^{1,2}(\Omega) \\ \|u\| = 1}} \|\nabla u\|^2)$ and corresponds to $c_1 = 1$, $c_k = 0$ for $k > 1$.

inductively,

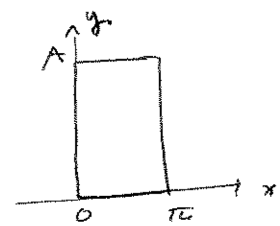
$$\lambda_k = \min_{\substack{u \perp \operatorname{span}\{f_1, f_2, \dots f_{k-1}\} \\ u \in W_0^{1,2}(\Omega) \\ \|u\|_{L^2} = 1}} \|\nabla u\|^2$$

This characterization leads to beautiful geometry.

ex one immediate consequence: If $\Omega_1 \subset \Omega_2$
then $\lambda_k(\Omega_1) \geq \lambda_k(\Omega_2)$

i.e. larger drumheads have lower natural frequencies with equal tension.

Example eigenbases of Δ on rectangle $0 \leq x \leq \pi$, $0 \leq y \leq A$



$\{ \sin nx \sin \frac{m\pi}{A} y \}_{m,n \in \mathbb{N}}$

are eigenfunctions, with Dirichlet data
and eigenvales $\lambda_{mn} = n^2 + \left(\frac{m\pi}{A}\right)^2$.

easy to check $\perp$, so up to ~~or~~ normalization do they give a complete basis ??

ans: Let $f \in C^1((0,\pi) \times (a,b)) \cap C([0,\pi] \times [a,b])$.

for each y, $f(x,y) = \sum_{n=1}^{\infty} b_n(y) \sin nx$ converges absolutely uniformly.

$b_n(y) = \frac{2}{\pi} \int_0^{\pi} f(x,y) \sin nx \, dx$, $\sum_n b_n^2(y) \leq M \quad \forall y$.

but $b_n(y)$ is itself differentiable in y,

so $b_n(y) = \sum_{m=1}^{\infty} b_{nm} \sin \frac{m\pi}{A} y$ conv. abs. unif.

$b_{nm} = \frac{2}{A} \int_0^A b_n(y) \sin \frac{m\pi}{A} y \, dy$

$\sum_m b_{nm}^2 \leq M$.

so $f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \sin nx \sin \frac{m\pi}{A} y$

$= \lim_{N \to \infty} \sum_{n,m=1}^{N} b_{nm} \sin nx \sin \frac{m\pi}{A} y$

converges uniformly absolutely.

If, for some eigenvalue, our claimed "basis" was deficient, there would be an eigenfunction $\perp$ to our collection and we wouldn't be able to approximate it uniformly using our collection because such approx $\Rightarrow L^2$ convergence $\divideontimes$.