Math 5440
Wed Dec. 1

Do the infinite dim'l proof for spectral theorem (Monday notes);
proof continues in today's notes...

Claim 2 on page 7 Monday is a consequence of

(sub) <u>Theorem</u>: Let $\{h_1, .. h_m\}$ be a basis for $\ker T$ (which we assumed finite dim'l).
Let $\{f_\ell\}_{\ell \in \mathbb{N}}$ be the orthonormal eigenfunctions we constructed inductively

$\perp$ ← orthog. complement

Then $\left(\overline{\text{span}\{f_\ell\}}_{\ell \in \mathbb{N}}\right)^{\perp} = \ker T.$

Thus $\overline{\text{span}\left(\{h_j\}_{j=1..m} \cup \{f_\ell\}_{\ell \in \mathbb{N}}\right)} = H.$

There is an important subtlety here:
For a <u>finite dim'l</u> $V \subset H$, with orthonormal basis $\{u_1, u_2, .. u_n\}$,
every $x \in H$ is written uniquely as

$$x = \text{proj}_V x + z$$

where $\text{proj}_V x = \sum_{j=1}^{n} \langle x, u_j \rangle u_j$ minimizes $|x - \tilde{v}|$ $\forall \tilde{v} \in V$
(is the nearest point)

and $z \in V^{\perp}$.

This existence of a unique orthogonal decomposition only holds for <u>infinite dimensional</u> $V \subset H$ if they are <u>closed</u> subspaces $\left(\{v_k\} \subset V,\ v_k \to x \in H \Rightarrow x \in V\right)$.
And the proof is interesting. (see next page.)
(Notice finite dim'l subspaces are automatically closed, but
$\text{span}\{f_\ell\}_{\ell \in \mathbb{N}}$ is <u>not</u> closed, which is why we take its closure.)

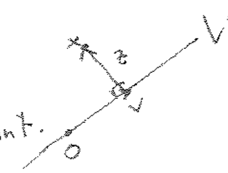
Lemma for (sub) Theorem

Let $V \subset H$ be a closed subspace of Hilbert space H.
Then $\forall\, x \in H$ $\exists!$ decomposition

$$x = v + z$$

with $v \in V$ and $z \in V^{\perp}$.

v is called $\text{proj}_V x$ and $|x-v| = \min\limits_{\tilde{v} \in V} |x-\tilde{v}|$



proof

recall $z \in V^{\perp} \Rightarrow |x-v| = \min\limits_{\tilde{v}\in V} |x-\tilde{v}|$ by Pythagorean Thm, and v is unique closest point.

but the converse is also true. Namely, if

$$|x-v| = \min_{\tilde{v}\in V} |x-\tilde{v}|$$

then $z = x - v \in V^{\perp}$: This is because

$F(\varepsilon) = |x-(v+\varepsilon\tilde{v})|^2$ has a minimum at $\varepsilon = 0$, so $F'(0)=0$

$$F(\varepsilon) = |(x-v) - \varepsilon\tilde{v}|^2$$
$$= |x-v|^2 - 2\varepsilon\langle x-v, \tilde{v}\rangle + \varepsilon^2|\tilde{v}|^2$$

$0 = F'(0) = -2\langle x-v, \tilde{v}\rangle = -2\langle z, \tilde{v}\rangle$ so $z \in V^{\perp}$.

So the lemma follows if we prove $\exists\, v \in V$ minimizing $|x-\tilde{v}|$.

So let $d := \inf\limits_{\tilde{v}\in V} |x-\tilde{v}|$

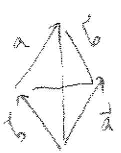
if $d=0$ then $x \in V$ (because V is closed) and x is the closest point in V to itself (and $z=0$).

if $d>0$ pick $\{v_j\} \subset V$
$$|x-v_j| \to d.$$

claim $\{v_j\}$ is Cauchy. If true then $\{v_j\} \to v \in H$ by completeness but $v \in V$ because V closed.

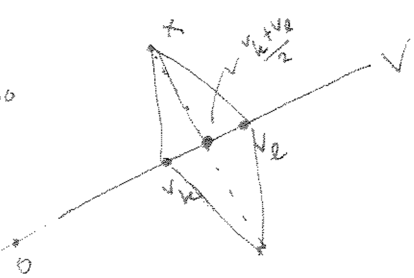
and $|x-v| = d$

parallelogram identity!



$|a+b|^2 + |a-b|^2$
$= 2|a|^2 + 2|b|^2$

So



$\frac{1}{4}(\square) =$

$$\underbrace{\left|x - \frac{v_k+v_\ell}{2}\right|^2}_{\geq d^2} + \left|\tfrac{1}{2}(v_k - v_\ell)\right|^2 = \underbrace{\tfrac{1}{2}|x-v_k|^2 + \tfrac{1}{2}|x-v_\ell|^2}_{\xrightarrow{k,\ell\to\infty}\ d^2}$$

$$\Downarrow \quad \left|\tfrac{1}{2}(v_k-v_\ell)\right|^2 \xrightarrow{k,\ell\to\infty} 0$$

We can now finish proof of (sub) Theorem, hence of spectral theorem:

Consider $\overline{\text{span}\{f_j\}_{j\in\mathbb{N}}}$. We claim it's orthogonal complement is $\ker T$.

Since orthogonal complements are automatically closed (we actually used this before)

$\left(\overline{\text{span}\{f_j\}_{j\in\mathbb{N}}}\right)^{\perp} := H_\infty$ is a Hilbert space, as when we took complements of finite dim'l subspaces.

Since T preserves $\text{span}\{f_j\}_{j\in\mathbb{N}}$
and is continuous
it also preserves its closure.

so, for $f \in \overline{\text{span}\{f_j\}}$
$w \in H_\infty$,
$0 = \langle Tf, w\rangle = \langle f, Tw\rangle \Rightarrow T: H_\infty \to H_\infty$ is compact and selfadjoint.

as before $\sup\limits_{w\in H_\infty} \frac{|\langle Tw,w\rangle|}{\langle w,w\rangle}$ is attained.

if this sup is positive and attained by $f_\infty \in H_\infty$
then $Tf_\infty = \lambda_\infty f_\infty$ $\lambda_\infty \neq 0$.
$\Rightarrow |\lambda_\infty| > |\lambda_n|$ for n sufficiently large
$\times$

Thus $\sup\limits_{w\in H_\infty} \frac{|\langle Tw,w\rangle|}{\langle w,w\rangle} = 0$

$\Rightarrow \|T|_{H_\infty}\|_{op} = 0 \Rightarrow H_\infty \subset \ker T$
$\Rightarrow H_\infty = \ker T$ since

$x \in H$
$\Rightarrow x = \underset{\overline{\text{span}\{f_j\}}}{\underset{\cap}{v}} + \underset{H_\infty}{\underset{\cap}{z}}$

and $T(v) = 0 \Rightarrow v = 0$

Application to PDE.

return to examples from page 2:

2) $T: L^2(\Omega) \to L^2(\Omega)$

$$Tf(x) = \int_\Omega G(x,\xi) f(\xi) d\xi$$

$G(x,\xi)$ = Green's fcn for Δ & Dirichlet data on Ω

$\ker T = 0$ $\quad \left(\langle Tf, f \rangle = \iint_{\Omega\,\Omega} G(x,\xi) f(x) f(\xi)\, dx\, d\xi \right.$

but $G(x,\xi) < 0$ in $\Omega \times \Omega$
by the maximum principle:
for fixed $\xi \in \Omega$,

$$\begin{cases} \Delta_x G(x,\xi) = 0 & x \neq \xi \\ G(x,\xi) = 0 & x \in \partial\Omega \\ G(x,\xi) \to -\infty & \text{as } x \to \xi. \end{cases}$$

so $\langle Tf, f \rangle \leq 0$
and $= 0$ iff
$f = 0$ in $L^2(\Omega)$.

if $Tf = \lambda f$, $|f| = 1$
then $\langle Tf, f \rangle = \lambda$
so all eigenvalues are negative.

the equation

$$* \quad \int_\Omega G(x,\xi) f(\xi) d\xi = \lambda f(x)$$

implies that if $f \in L^2(\Omega)$ then $Tf \in W^{2,2}(\Omega)$
then $f \in W^{2,2}(\Omega) \Rightarrow Tf \in W^{4,2}(\Omega)$
$\Rightarrow \ldots f \in W^{k,2}(\Omega)$ $\forall k$
$\Rightarrow f \in C^\infty(\Omega)$.

1st & 2nd "weak" derivatives are in L^2

"Sobolev spaces"

(see 6.??? level PDE course)

also for the proof that T is compact which follows from the fact that $W^{2,2}(\Omega)$ is a compact subspace of $L^2(\Omega)$.

in particular can take actual Δ of $*$

$\Rightarrow f(x) = \lambda \Delta f(x)$

$\Rightarrow \Delta f(x) = \frac{1}{\lambda} f(x)$

So Spectral theorem implies

$\exists$ o.n. complete (Hilbert) basis of Dirichlet eigenfunctions, $\{f_k\}$

$\Delta f_k = -\lambda_k f_k$ in Ω $\quad \lambda_k \to \infty$.

We'll do example 3) (the type of ODE's we get from separating variables) in more detail Friday