Math 5440
Monday Aug 30

We'll finish chapter 1 sometime on Wed. (notes are all here)
Then review divergence theorem.
Then derive a fairly broad collection of the classical PDE's, before continuing with text.

- Recall the soln to homog W.E. IVP

$$\text{IVP}\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

is $u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(r)\,dr$ (and is unique)

- Recall that to solve IBVP on interval $0 \leq x \leq l$ with

$$\text{BC}\begin{cases} u(0,t) = 0 & t \geq 0 \\ u(l,t) = 0 \end{cases}$$

simply extend f & g to be odd wrt $x=0$ & $x=l$, to $2l$-periodic functions. (and a quicker way to show the resulting u is also odd wrt $x=0$ & $x=l$, in particular that it satisfies $u(0,t) = u(l,t) = 0\ \forall t$, although this clear directly), is to use the Hw problem that

$\psi(x)$ odd (resp even) iff ψ' even (resp odd),

since $u_x = \frac{1}{2}\left(f'(x+ct) + f'(x-ct)\right) + \frac{1}{2c}\left(g(x+ct) - g(x-ct)\right)$

solns to this IBVP are unique since any $u(x,t)$ solving (IVP) & (BC) can be extended to $\mathbb{R}^{2+}$ so as to be odd wrt $x=0$, $x=l$ & $2l$-periodic, and it will satisfy the $\mathbb{R}^{2+}$ IVP, with f, g equal to the odd extensions of f, g.

- Another natural BC is

$$\text{BC}\begin{cases} u_x(0,t) = 0 \\ u_x(l,t) = 0 \end{cases}$$

this is called the "free" BC, and for a transverse spring oscillation would correspond to no vertical force at $x=0$ or $x=l$, so the spring is free to move transversely at the endpoints.

this problem is solved uniquely using even reflections of f, g wrt $x=0$ & $x=l$, also creating $2l$ period fcns, and unique solution u which is also even wrt $x=0, l$.

- You can also consider one fixed and one free end.
  See hw problems.

Maple & slinkies illustrate this.

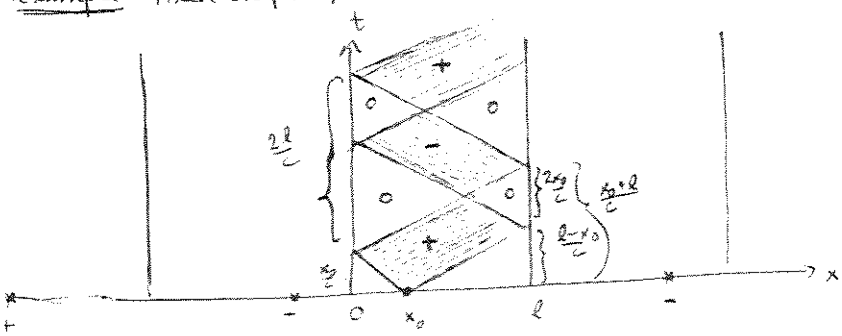
§1.2–1.4

We can also consider domains of influence and dependence for the natural IBVP's on $0 \le x \le l$

<u>example</u> fixed endpoint $u(0,t) = 0 = u(l,t) \quad t \ge 0$

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(r)\,dr$$

odd ~~even~~ extension



domain of influence for x_0
the edges result from changing f near x_0 (& odd extension)
the interiors result from changing g near x_0 (& odd extension).

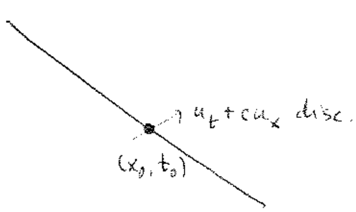
domain of influence looks different for free or mixed endpoint BC's → because of even vs. odd extension!
– see HW.

- A non-C^2 function which is a <u>uniform limit</u> of C^2 sol'ns to the (homogeneous) wave equation is called a <u>generalized solution</u> p. 15.

  ↑ do you recall this def?

  Interesting examples are when the initial value & veloc. fcns f and g are continuous and piecewise C^2 as for the tent function.
  Points where directional derivatives are discontinuous are interesting and have interesting propogation properties – they move along characteristics
  – see §3 & HW.

<u>example</u> suppose u is a generalized sol'n with $u_t + cu_x$ discontinuous at (x_0, t_0)
then this discontinuity persists along the characteristic thru (x_0, t_0) in the <u>other</u> characteristic dir $\langle -c, 1 \rangle$



<u>pf</u> $\quad u_t + cu_x = \frac{1}{2}\left(2cf(x+ct) + 2g(x+ct)\right)$

is invariant if (x,t) is changed by any multiple of $\langle -c, 1 \rangle$.

∎

§1.5 The solution to non-homogeneous IVP on $\mathbb{R}^{2+}$

(with corresponding odd/even extension sol'ns to IBVP's)

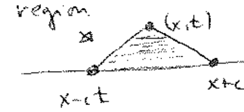
Theorem 1

$$\text{IVP}\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

has unique sol'n.

$$u(x,t) = \underbrace{\frac{1}{2c}\int_{x-ct}^{x+ct} g(\tilde{x})\,d\tilde{x}}_{\text{solves IVP with RHS data } \begin{bmatrix}0\\0\\g\end{bmatrix}} + \underbrace{\frac{1}{2}\big(f(x+ct) - f(x-ct)\big)}_{\text{solves IVP with RHS data } \begin{bmatrix}0\\f\\0\end{bmatrix}} + \underbrace{\frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\tilde{x},\tau)\,d\tilde{x}\,d\tau}_{\text{solves IVP with RHS data } \begin{bmatrix}F\\0\\0\end{bmatrix}}$$

region



proof : Sol'n unique because difference of any two sol'ns satisfies IVP with $F=0$, $f=g=0$, so by previous results must be the zero fcn.

So it suffices to show the third piece of this claim, that solves

$$* \begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & x\in\mathbb{R},\ t \geq 0 \\ u(x,0) = 0 & x \in \mathbb{R} \\ u_t(x,0) = 0 & x \in \mathbb{R}. \end{cases}$$

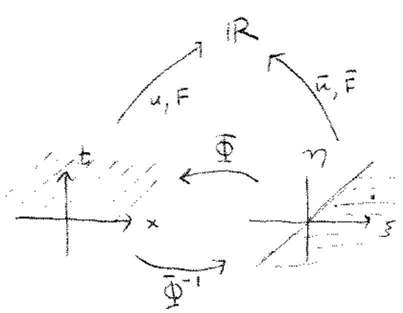
Use previous COV

$$\Phi(\xi,\eta) = \begin{bmatrix} \frac{1}{2}(\xi+\eta) \\ \frac{1}{2c}(\xi-\eta) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Phi^{-1}(x,t) = \begin{bmatrix} x+ct \\ x-ct \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

$\bar{u} = u \circ \Phi \qquad \bar{F} = F \circ \Phi$

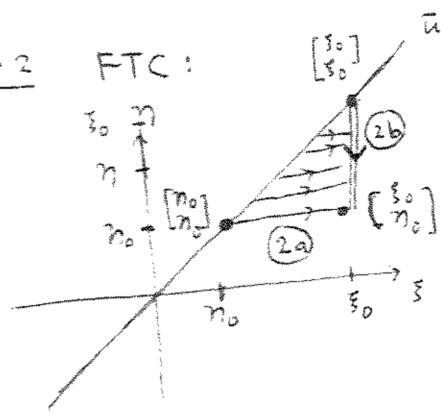
$u = \bar{u} \circ \Phi^{-1} \qquad F = \bar{F} \circ \Phi^{-1}$



Step 1 $*$ transforms to

$$\begin{cases} \bar{u}_{\xi\eta} = -\frac{1}{4c^2}\bar{F} & \xi \geq \eta \\ \bar{u}(x,x) = 0 & \\ \bar{u}_\xi(x,x) = 0 & x \in \mathbb{R} \\ \bar{u}_\eta(x,x) = 0 & \end{cases}$$

Step 2 FTC:



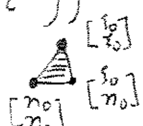
(2a) $\bar u_\eta(\xi_0,\eta) = \cancel{\bar u_\eta(\eta,\eta)}^{\,0} + \int_\eta^{\xi_0} \bar u_{\eta\xi}\,d\xi$

(2b) $\bar u(\xi_0,\eta_0) = \cancel{\bar u(\xi_0,\xi_0)}^{\,0} + \int_{\xi_0}^{\eta_0} \bar u_\eta(\xi_0,\eta)\,d\eta = \int_{\xi_0}^{\eta_0}\left(\int_\eta^{\xi_0} \bar u_{\eta\xi}\,d\xi\right)d\eta = -\int_{\eta_0}^{\xi_0}\int_\eta^{\xi_0} \bar u_{\eta\xi}\,d\xi\,d\eta$

is a double integral over the triangle $\eta_0 \le \eta \le \xi_0$ & for fixed η, $\eta \le \xi \le \xi_0$.

Step 3 So by PDE

$$\bar u(\xi_0,\eta_0) = \frac{1}{4c^2}\iint \bar F(\xi,\eta)\,dA$$



Step 4 COV in double integral

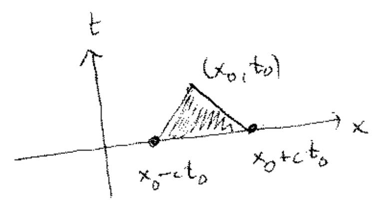
is Jacobian matrix.

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix} = \underline{\Phi}(x,t) = \begin{bmatrix}x+ct\\ x-ct\end{bmatrix} = \begin{bmatrix}1 & c\\ 1 & -c\end{bmatrix}\begin{bmatrix}x\\ t\end{bmatrix}$$

$\frac{\partial(\xi,\eta)}{\partial(x,t)} = |-2c| = 2c$ area expansion factor (abs of Jacobian det.)

$\xi_0 = x_0 + ct_0$
$\eta_0 = x_0 - ct_0$

so ξ-η triangle is Φ^{-1} image of



therefore

$$u(x_0,t_0) = \frac{1}{4c^2}\iint F(x,t)\,2c\,dA = \frac{1}{2c}\iint F\,dA \quad ∎$$

Theorem 2 To solve fixed or free endpoint IBVP's for the inhomogeneous wave equation just use odd or even extension of F, in analogy to how you did it for f, g.