Math 5440
Wed. Aug 25
§1.1–1.2

Pick up where we left off on Monday, ≈ page 3 of those notes.
We were considering homog. wave eqtn

$$* \quad \boxed{u_{tt} - c^2 u_{xx} = 0}$$

with constant "speed" c.
We showed that if $U = U(z)$ was any C^2 fcn of one variable, then

$$u(x,t) = U(x - ct)$$
$$u(x,t) = U(x + ct)$$ solve *.

before we play with slinky, prove

<u>Theorem</u> let $u(x,t)$ solve * in $\mathbb{R}^2$, or in the upper half plane $t \geq 0$, $u \in C^2$.
then $\exists$ functions p, q s.t.

$$u(x,t) = p(x+ct) + q(x-ct)$$

so every soln to * is a superposition of a wave traveling to the left with one traveling to the right, both at speed c.

<u>proof</u> Using partial differential operators we may rewrite * as

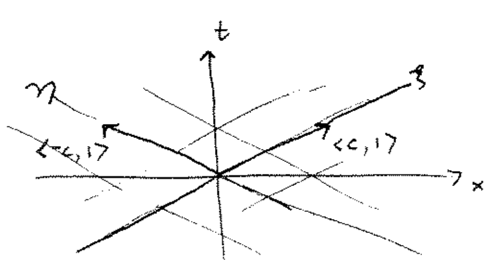
$$L(u) = \partial_t \circ \partial_t u - c^2 \partial_x \circ \partial_x u = 0$$

the operators "factor" (because c is a constant)

$$(\partial_t - c\partial_x) \circ (\partial_t + c\partial_x)(u) = 0$$ because $u_{xt} = u_{tx}$

(or in reverse order).

i.e. $-cu_x + u_t = 0$ along lines with direction vector $\langle -c, 1\rangle$
$cu_x + u_t = 0$ " " " $\langle c, 1\rangle$



this suggests using new coord system in $\mathbb{R}^2$, so that (ξ, η) are coords wrt basis
$\left\{ \begin{bmatrix} c \\ 1 \end{bmatrix}, \begin{bmatrix} -c \\ 1 \end{bmatrix} \right\}$

i.e. $\begin{bmatrix} c & -c \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} x \\ t \end{bmatrix}$

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{1}{2c} \begin{bmatrix} 1 & c \\ -1 & c \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = \frac{1}{2c} \begin{bmatrix} x + ct \\ -x + ct \end{bmatrix}$$

we choose $\xi = x + ct$, $\eta = x - ct$ instead, for convenience

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix} = \begin{bmatrix}1 & c\\ 1 & -c\end{bmatrix}\begin{bmatrix}x\\ t\end{bmatrix} \quad ; \quad \begin{bmatrix}x\\ t\end{bmatrix} = -\frac{1}{2c}\begin{bmatrix}-c & -c\\ -1 & 1\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2c} & -\frac{1}{2c}\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix}$$

thus $u_\xi = u_x x_\xi + u_t t_\xi = \frac{1}{2}u_x + \frac{1}{2c}u_t = \frac{1}{2c}(u_t + c u_x)$ $\qquad \partial_\xi = \frac{1}{2c}(\partial_t + c\partial_x)$

$u_\eta = u_x x_\eta + u_t t_\eta = \frac{1}{2}u_x - \frac{1}{2c}u_t = -\frac{1}{2c}(u_t - c u_x)$ $\qquad \partial_\eta = -\frac{1}{2c}(\partial_t - c\partial_x)$

so $*$ becomes

$**$ $\boxed{u_{\xi\eta} = 0}$ $\qquad$ (technically, $\partial_\eta \circ \partial_\xi \big(\underbrace{u(x(\xi,\eta), t(\xi,\eta))}_{\tilde{u}(\xi,\eta)}\big) = 0$

$$\partial_\eta(\partial_\xi \tilde{u}) = 0 \iff \partial_\xi(\tilde{u}) = P(\xi) \quad \text{(no } \eta \text{ dependence)}$$

$$\Rightarrow \tilde{u}(\xi,\eta) = \underbrace{\tilde{u}(0,\eta)}_{q(\eta)} + \underbrace{\int_0^\xi P(s)\,ds}_{p(\xi)}$$

so $u(x,t) = q(x-ct) + p(x+ct)$ ■

ho play with slinky.

notice how you can solve IBVP's on $\overset{x}{[0,\ell)} \times \overset{t}{[0,\infty)} = \{(x,t) \text{ s.t. } 0 \le x \le \ell \text{ and } t \ge 0\}$

by superposition of traveling waves for $x \in \mathbb{R}$.

Then finish derivation of longitudinal wave eqtn. on Monday notes.

If time, p. 3 today.

Sol'n to

$$u_{tt} - c^2 u_{xx} = 0 \qquad (c \text{ const}) \quad x \in \mathbb{R},\ t \geq 0$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

(can be adapted to IBVP on $0 \le x \le \ell$ by odd reflection of f, g across $x=0$ & $x=\ell$.)

$$u(x,t) = p(x+ct) + q(x-ct)$$

@ $t=0$: $f(x) = p(x) + q(x) \Rightarrow f' = p' + q'$
$g(x) = c\,(p'(x) - q'(x))$

$$\begin{bmatrix} 1 & 1 \\ c & -c \end{bmatrix}\begin{bmatrix} p' \\ q' \end{bmatrix} = \begin{bmatrix} f' \\ g \end{bmatrix} \qquad \begin{bmatrix} p' \\ q' \end{bmatrix} = -\frac{1}{2c}\begin{bmatrix} -c & -1 \\ -c & 1 \end{bmatrix}\begin{bmatrix} f' \\ g \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2c} \\ \frac{1}{2} & -\frac{1}{2c} \end{bmatrix}\begin{bmatrix} f' \\ g \end{bmatrix}.$$

$$p' = \tfrac{1}{2} f' + \tfrac{1}{2c} g \Rightarrow p(\xi) = \tfrac{1}{2} f(\xi) + \tfrac{1}{2c}\int_0^{\xi} g(\bar{x})\,d\bar{x} + K_1$$
$$q' = \tfrac{1}{2} f' - \tfrac{1}{2c} g \Rightarrow q(\eta) = \tfrac{1}{2} f(\eta) - \tfrac{1}{2c}\int_0^{\eta} g(\bar{x})\,d\bar{x} + K_2$$

since $p(0) + q(0) = f(0)$
$K_1 + K_2 = 0.$

So
$$\boxed{u(x,t) = \tfrac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}}$$