Mathematics 5440-1
Fall 2010

Class Time and Place: M, W, F, 10:45-11:35, LCB 204

Class website http://www.math.utah.edu/~korevaar/5440fall10

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Office Hours: M, W 2:20-3:00; Th 9:30-10:30, and by appointment
Problem Session: either W 6:30-8:00 p.m. or Th 12:30-2:00, to be determined by class preference. Location to be announced.

Text: *A First Course in Partial Differential Equations, with Complex Variables and Transform Methods,* by H.F. Weinberger ISBN 0-486-68640-X

Prerequisites: Math 3210-3220 or equivalent; we will use concepts from single and differential and integral multivariable analysis such as continuity, differentiability, inverse function theorem, function convergence theorems and the vector calculus theorems. Although the formal prerequisite requirement is a grade of at least *C* in 3220, you will probably have difficulty with the 5440 course material if you don't understand the analysis material at an *A* or *B* level. You will be expected to learn the key theorems and derivations in this course, and your homework will include theoretical problems along with computations and applications.

Course Description: This course is an introduction to the methods used to study of **p**artial **d**ifferential **e**quations. PDEs and systems of PDEs arise in diverse areas of science and mathematics, and can have quite different characteristic behaviors, as you would expect by the diversity of dynamical and static systems they can be used to model. The core of the 5440 course material is Chapters 1-7 of the Weinberger text, i.e. a rigorous study of the basic partial differential equations of classical physics and mathematics: the wave, heat and Laplace equations. We will study derivations of these equations, along with the classical methods for solving and estimating the solutions to the associated natural initial and/or boundary value problems. These methods include the method of characteristics, integral forms of solution including Green's functions, maximum principles, Fourier series solutions and generalizations. As appropriate I will provide supplementary material from other texts or on-line resources, in order to fit this core material into the larger context of the more general study of PDEs. There is currently a wealth of interesting material easily available on-line.

I will pass a sheet around on the first day of class so that you may each pick a 20 minute interval to visit me in my office and get acquainted. I'm interested in knowing your backgrounds and expectations for this course, and in answering questions you may have about it.

Grading: There will be two midterms, a comprehensive final examination, and homework. Each midterm will count for 20% of your grade, homework will count for 30%, and the final exam will make up the remaining 30%. All exams will be given in our classroom. The midterm exam dates are **Monday October 4** and **Monday November 15**. The final exam is at the University time and date of **Tuesday December 14, 10:30-12:30.**

You may opt out of the final exam by completing an individual or 2-person group project on some application or extension of the course material. Each project shall consist of a 5-

10 page expository paper, and a presentation to the class of at least 20 minutes in length, but possibly longer. I will be available for pre-presentation consultation and practice. Project groups and topics must be approved by me, **by Friday November 5**.

Although I don't recommend it, some graduate students from departments other than Math may need to enroll in the parallel course to this one, Math 6850. If you are one of these students you will be required to complete both a project (at an appropriately advanced level) and the in class final exam. Luckily, since PDEs arise in so many areas of research, there may be a topic which ties in very nicely with your graduate interests.

Homework assigned at successive FMW lectures will be collected on the immediately following Friday, and will be graded. Note that in addition to office hours you may attend the weekly problem/discussion session. Work individually and collaboratively, but each student is responsible for handing in original (non-copied) work.

http://www.math.utah.edu/~schmitt/pde5440-09.pdf

4 Additional texts

1. N. Asmar: Partial Differential Equations and Boundary Value Problems, Prentice Hall, Upper Saddle River, NJ, 2000. (Similar level.)

2. R. Courant and D. Hilbert: Methods of Mathematical Physics, vols. 1 and 2, Wiley-Interscience, New York, 1962. (More advanced, a classic!)

3. C. Edwards and D. Penny: Differential Equations and Boundary Value Problems, Prentice Hall, Upper Saddle River, NJ, 1996. (More elementary.)

4. P. Garabedian: Partial Differential Equations, Wiley, New York, 1964. (More advanced.)

5. R. Haberman. Elementary Applied Partial Differential Equations, Prentice Hall, Upper Saddle River, NJ, 1998. (More elementary.)

6. F. John: Partial Differential Equations, Springer, New York, 1982. (More advanced.)

7. D. Logan: Applied Partial Differential Equations, Springer, New York, 1998. (Similar level.)

8. M. Pinsky: Introduction to Partial Differential Equations with Applications, Mc-Graw Hill, New York, 1984.

9. W. Strauss: Partial Differential Equation an Introduction, Wiley, New York, 1992. (Similar level; this text appeared in a new edition recently)

10. J. Troutman: Boundary Value Problems of Applied Mathematics, PWS Publishing, Boston, 1994. (More elementary.)

11. E. Zauderer: Partial Differential Equations of Applied Mathematics, 3rd edition, Wiley, Hoboken, 2006. (Similar level.)

is given and

$$u : U \to \mathbb{R}$$

is the unknown.

We *solve* the PDE if we find all u verifying (1), possibly only among those functions satisfying certain auxiliary boundary conditions on some part Γ of ∂U. By finding the solutions we mean, ideally, obtaining simple, explicit solutions, or, failing that, deducing the existence and other properties of solutions.

DEFINITIONS.

(i) *The partial differential equation* (1) *is called* linear *if it has the form*

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x)$$

for given functions a_α ($|\alpha| \leq k$), f. *This linear PDE is* homogeneous *if* $f \equiv 0$.

(ii) *The PDE* (1) *is* semilinear *if it has the form*

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0.$$

(iii) *The PDE* (1) *is* quasilinear *if it has the form*

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \ldots, Du, u, x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0.$$

(iv) *The PDE* (1) *is* fully nonlinear *if it depends nonlinearly upon the highest order derivatives.*

A *system* of partial differential equations is, informally speaking, a collection of several PDE for several unknown functions.

DEFINITION. *An expression of the form*

$$\mathbf{F}(D^k\mathbf{u}(x), D^{k-1}\mathbf{u}(x), \ldots, D\mathbf{u}(x), \mathbf{u}(x), x) = \mathbf{0} \quad (x \in U) \tag{2}$$

is called a k^{th}-order system of partial differential equations, *where*

$$\mathbf{F} : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \cdots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times U \to \mathbb{R}^m$$

is given and

$$\mathbf{u} : U \to \mathbb{R}^m, \ \mathbf{u} = (u^1, \ldots, u^m)$$

is the unknown.

Here we are supposing that the system comprises the same number m of scalar equations as unknowns $(u^1, \ldots, u^m)$. This is the most common circumstance, although other systems may have fewer or more equations than unknowns. Systems are classified in the obvious way as being linear, semilinear, etc.

from Partial Differential Equations, 2nd edition by L.C. Evans

NOTATION. We write "PDE" as an abbreviation for both the singular "partial differential equation" and the plural "partial differential equations".

1.2. EXAMPLES

There is no general theory known concerning the solvability of all partial differential equations. Such a theory is extremely unlikely to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modeled by PDE. Instead, research focuses on various particular partial differential equations that are important for applications within and outside of mathematics, with the hope that insight from the origins of these PDE can give clues as to their solutions.

Following is a list of many specific partial differential equations of interest in current research. This listing is intended merely to familiarize the reader with the names and forms of various famous PDE. To display most clearly the mathematical structure of these equations, we have mostly set relevant physical constants to unity. We will later discuss the origin and interpretation of many of these PDE.

Throughout $x \in U$, where U is an open subset of $\mathbb{R}^n$, and $t \geq 0$. Also $Du = D_x u = (u_{x_1}, \ldots, u_{x_n})$ denotes the gradient of u with respect to the spatial variable $x = (x_1, \ldots, x_n)$. The variable t always denotes time.

1.2.1. Single partial differential equations.

a. Linear equations.

1. *Laplace's equation*

$$\Delta u = \sum_{i=1}^n u_{x_i x_i} = 0.$$

2. *Helmholtz's (or eigenvalue) equation*

$$-\Delta u = \lambda u.$$

3. *Linear transport equation*

$$u_t + \sum_{i=1}^n b^i u_{x_i} = 0.$$

4. *Liouville's equation*

$$u_t - \sum_{i=1}^n (b^i u)_{x_i} = 0.$$

5. *Heat (or diffusion) equation*

$$u_t - \Delta u = 0.$$

6. *Schrödinger's equation*

$$iu_t + \Delta u = 0.$$

7. *Kolmogorov's equation*

$$u_t - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$

8. *Fokker–Planck equation*

$$u_t - \sum_{i,j=1}^{n} (a^{ij} u)_{x_i x_j} - \sum_{i=1}^{n} (b^i u)_{x_i} = 0.$$

9. *Wave equation*

$$u_{tt} - \Delta u = 0.$$

10. *Klein–Gordon equation*

$$u_{tt} - \Delta u + m^2 u = 0.$$

11. *Telegraph equation*

$$u_{tt} + 2du_t - u_{xx} = 0.$$

12. *General wave equation*

$$u_{tt} - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$

13. *Airy's equation*

$$u_t + u_{xxx} = 0.$$

14. *Beam equation*

$$u_{tt} + u_{xxxx} = 0.$$

b. Nonlinear equations.

1. *Eikonal equation*

$$|Du| = 1.$$

2. *Nonlinear Poisson equation*

$$-\Delta u = f(u).$$

3. *p-Laplacian equation*

$$\operatorname{div}(|Du|^{p-2} Du) = 0.$$

4. *Minimal surface equation*

$$\operatorname{div}\left(\frac{Du}{(1 + |Du|^2)^{1/2}}\right) = 0.$$

5. *Monge–Ampère equation*

$$\det(D^2 u) = f.$$

6. *Hamilton–Jacobi equation*

$$u_t + H(Du, x) = 0.$$

7. *Scalar conservation law*

$$u_t + \operatorname{div} \mathbf{F}(u) = 0.$$

8. *Inviscid Burgers' equation*

$$u_t + uu_x = 0.$$

9. *Scalar reaction-diffusion equation*

$$u_t - \Delta u = f(u).$$

10. *Porous medium equation*

$$u_t - \Delta(u^\gamma) = 0.$$

11. *Nonlinear wave equation*

$$u_{tt} - \Delta u + f(u) = 0.$$

12. *Korteweg–deVries (KdV) equation*

$$u_t + uu_x + u_{xxx} = 0.$$

13. *Nonlinear Schrödinger equation*

$$iu_t + \Delta u = f(|u|^2)u.$$

1.2.2. Systems of partial differential equations.

a. Linear systems.

1. *Equilibrium equations of linear elasticity*

$$\mu\Delta\mathbf{u} + (\lambda+\mu)D(\operatorname{div}\mathbf{u}) = \mathbf{0}.$$

2. *Evolution equations of linear elasticity*

$$\mathbf{u}_{tt} - \mu\Delta\mathbf{u} - (\lambda+\mu)D(\operatorname{div}\mathbf{u}) = \mathbf{0}.$$

3. *Maxwell's equations*

$$\begin{cases} \mathbf{E}_t = \operatorname{curl}\mathbf{B} \\ \mathbf{B}_t = -\operatorname{curl}\mathbf{E} \\ \operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{E} = 0. \end{cases}$$

b. Nonlinear systems.

1. *System of conservation laws*

$$\mathbf{u}_t + \operatorname{div}\mathbf{F}(\mathbf{u}) = \mathbf{0}.$$

2. *Reaction-diffusion system*

$$\mathbf{u}_t - \Delta\mathbf{u} = \mathbf{f}(\mathbf{u}).$$

3. *Euler's equations for incompressible, inviscid flow*

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot D\mathbf{u} = -Dp \\ \operatorname{div}\mathbf{u} = 0. \end{cases}$$

4. *Navier–Stokes equations for incompressible, viscous flow*

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot D\mathbf{u} - \Delta\mathbf{u} = -Dp \\ \operatorname{div}\mathbf{u} = 0. \end{cases}$$

See Zwillinger [**Zw**] for a much more extensive listing of interesting PDE.

1.3. STRATEGIES FOR STUDYING PDE

As explained in §1.1 our goal is the discovery of ways to solve partial differential equations of various sorts, but—as should now be clear in view of the many diverse examples set forth in §1.2—this is no easy task. And indeed the very question of what it means to "solve" a given PDE can be subtle, depending in large part on the particular structure of the problem at hand.

1.3.1. Well-posed problems, classical solutions.

The informal notion of a well-posed problem captures many of the desirable features of what it means to solve a PDE. We say that a given problem for a partial differential equation is *well-posed* if

(i) the problem in fact has a solution;

(ii) this solution is unique;

and

(iii) the solution depends continuously on the data given in the problem.

The last condition is particularly important for problems arising from physical applications: we would prefer that our (unique) solution changes only a little when the conditions specifying the problem change a little. (For many problems, on the other hand, uniqueness is not to be expected. In these cases the primary mathematical tasks are to classify and characterize the solutions.)

Now clearly it would be desirable to "solve" PDE in such a way that (i)–(iii) hold. But notice that we still have not carefully defined what we mean by a "solution". Should we ask, for example, that a "solution" u must be real analytic or at least infinitely differentiable? This might be desirable, but perhaps we are asking too much. Maybe it would be wiser to require a solution of a PDE of order k to be at least k times continuously differentiable. Then at least all the derivatives which appear in the statement of the PDE will exist and be continuous, although maybe certain higher derivatives will not exist. Let us informally call a solution with this much smoothness a *classical* solution of the PDE: this is certainly the most obvious notion of solution.

So by solving a partial differential equation in the classical sense we mean if possible to write down a formula for a classical solution satisfying (i)–(iii) above, or at least to show such a solution exists, and to deduce various of its properties.

1.3.2. Weak solutions and regularity.

But can we achieve this? The answer is that certain specific partial differential equations (e.g. Laplace's equation) can be solved in the classical sense, but many others, if not most others, cannot. Consider for instance