Geodesic Connectivity in Diffusion Tensor Imaging

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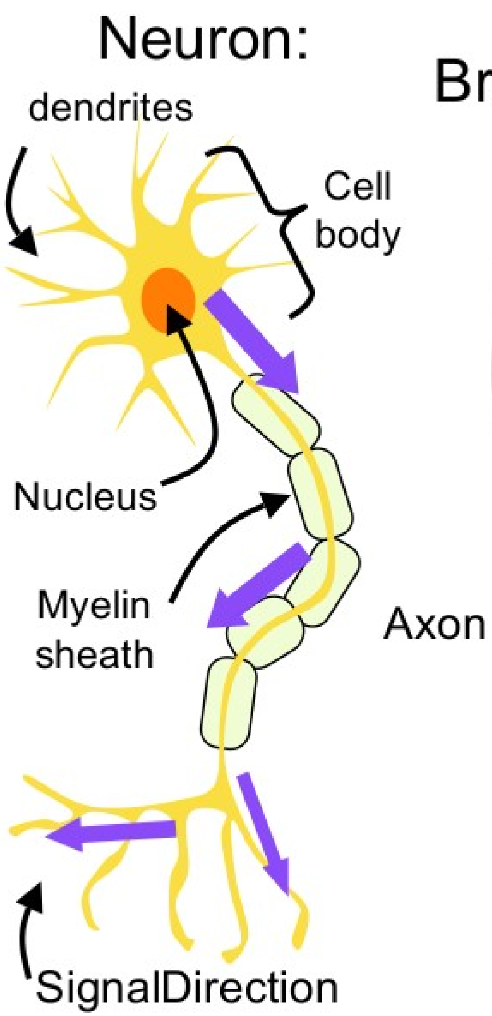
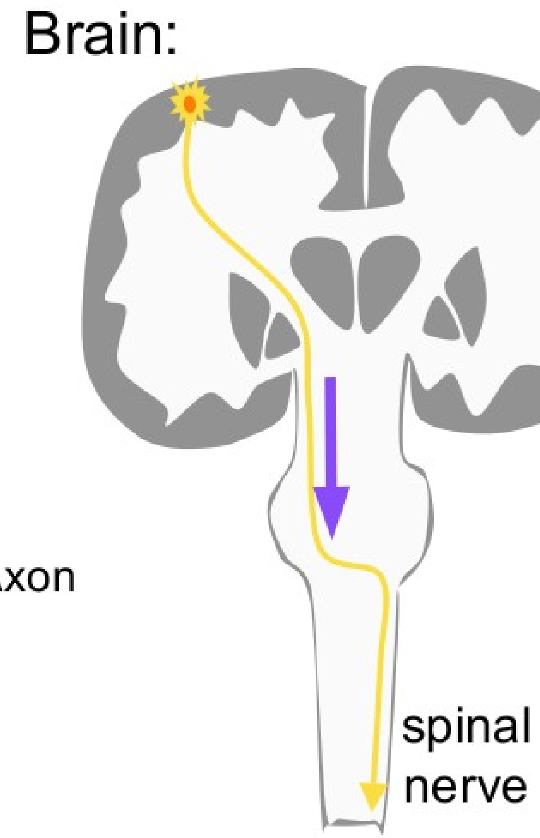
Underlying Biology[1]





- Gray matter (cortex + nuclei): cell bodies
- White matter: axons
- Myelin sheath speeds signal conduction
- Axon + sheath = nerve fibers
- Major white matter pathways aggregate many fibers into bundles

Diffusion Tensor Imaging

Diffusion tensor imaging(DTI) is a medical imaging modality that can reveal directional information in vivo in fibrous structures such as white matter or muscles.

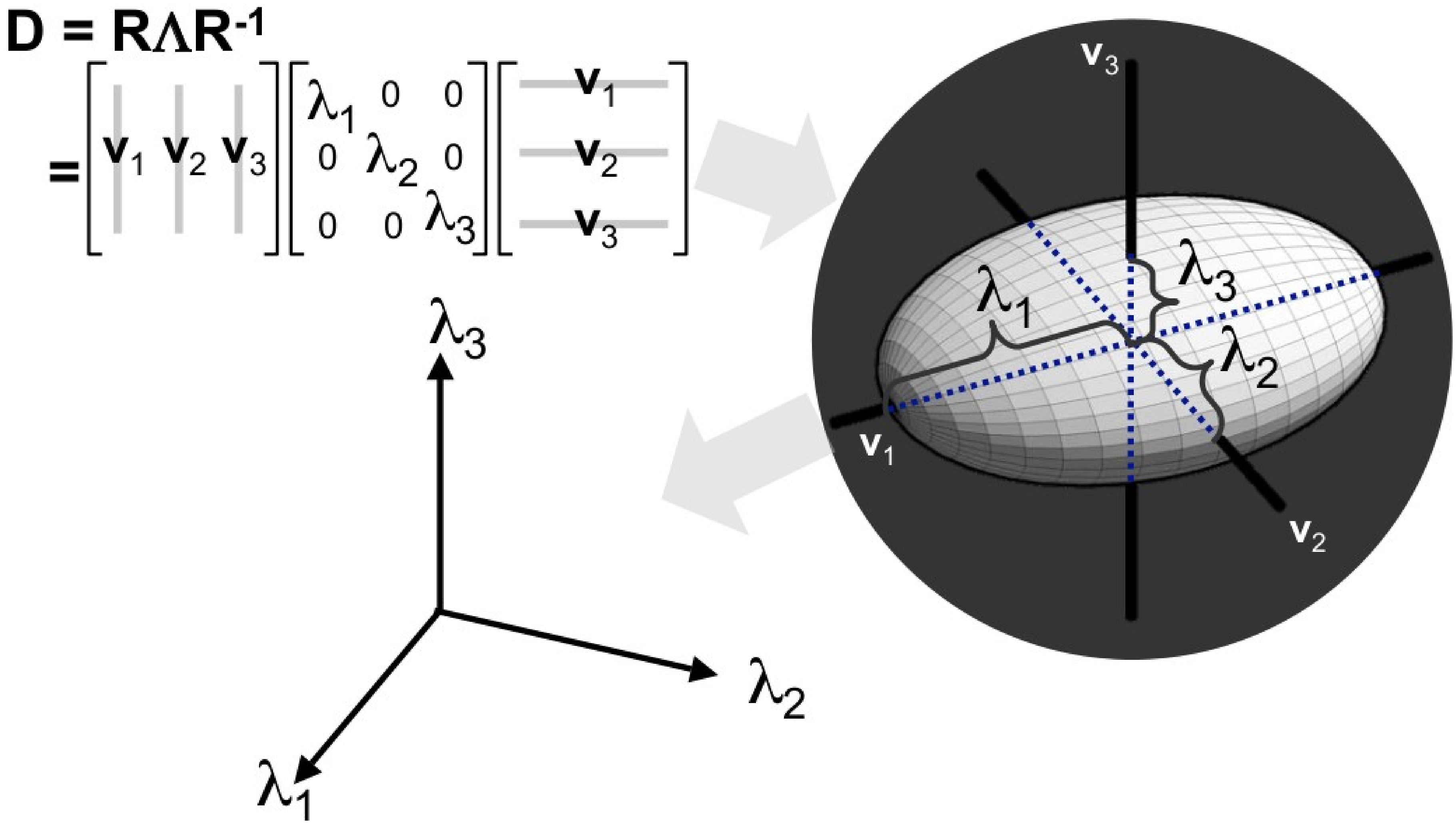
Each point of a DTI is a diffusion tensor, which measures a 3D diffusion process and has six interrelated tensor components.

3D Diffusion Tensor

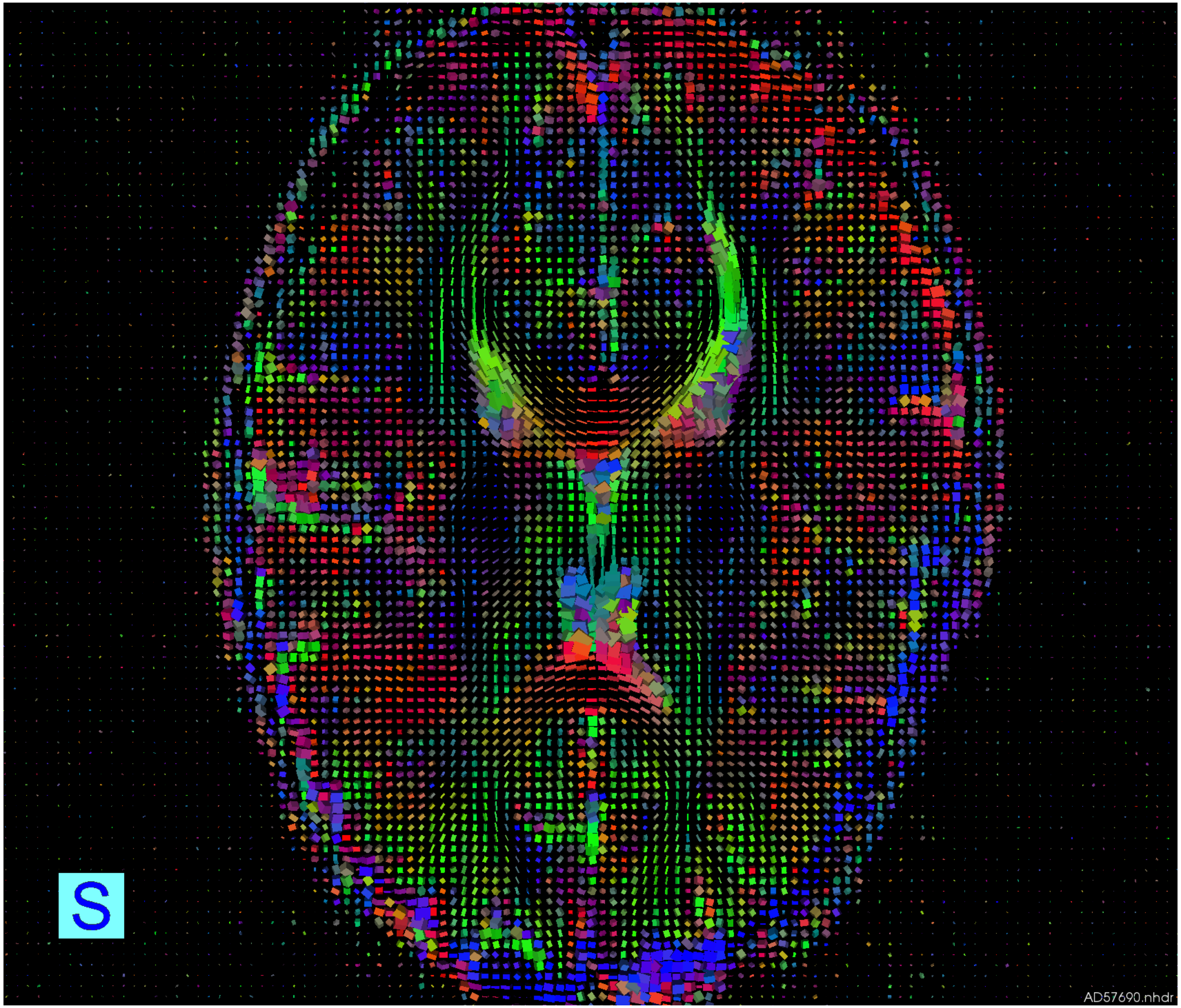
A 3D diffusion tensor is a 3x3 positive symmetric matrix

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_{xx} & \mathbf{D}_{xy} & \mathbf{D}_{xz} \\ \mathbf{D}_{xy} & \mathbf{D}_{yy} & \mathbf{D}_{yz} \\ \mathbf{D}_{xz} & \mathbf{D}_{yz} & \mathbf{D}_{zz} \end{bmatrix} \qquad D = R\Sigma R^T$$

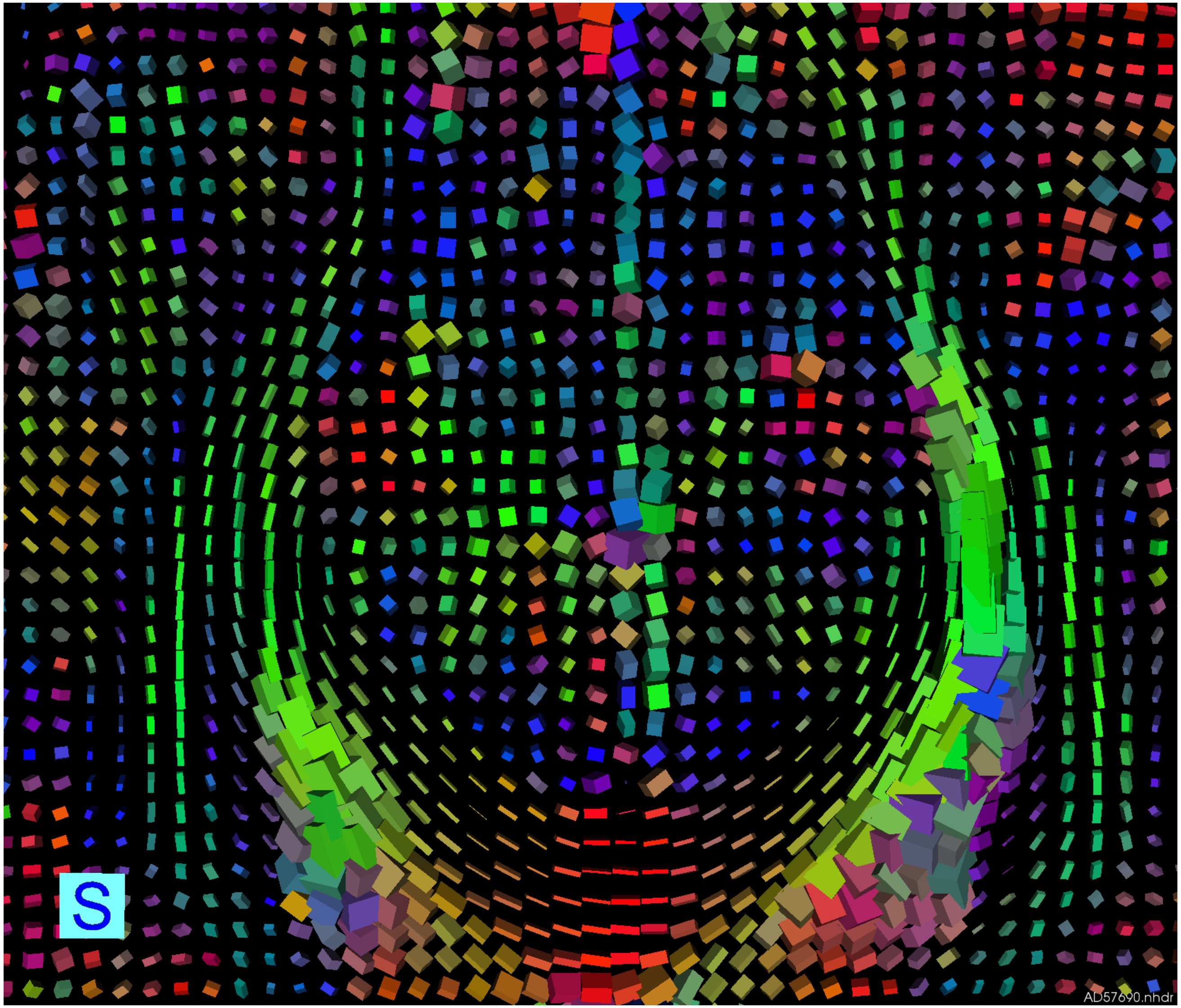
EigenDecomposition[1]



A Slice of DTI



Detail



Brain Connectivity

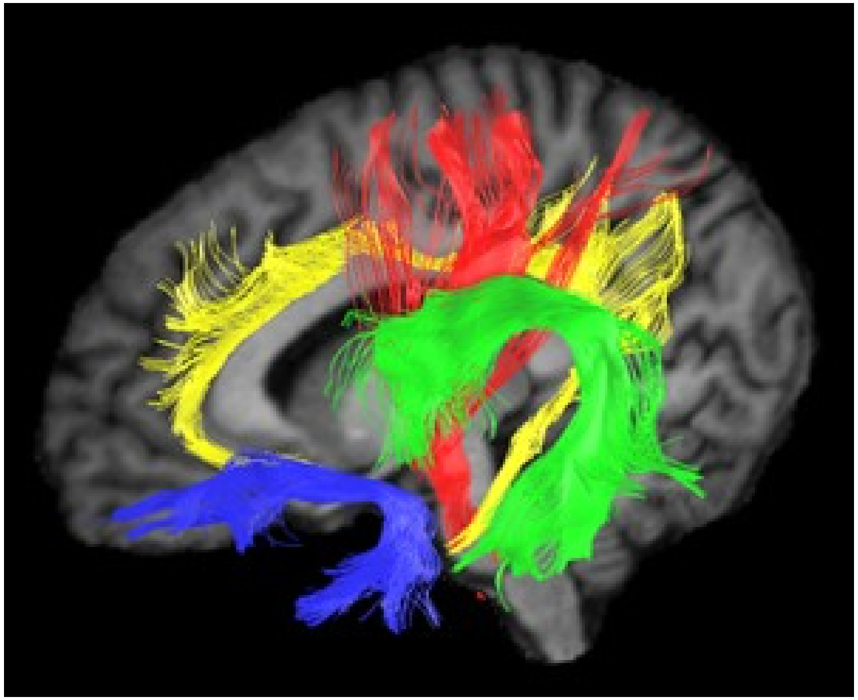
DTI can provide information about connections among brain regions.

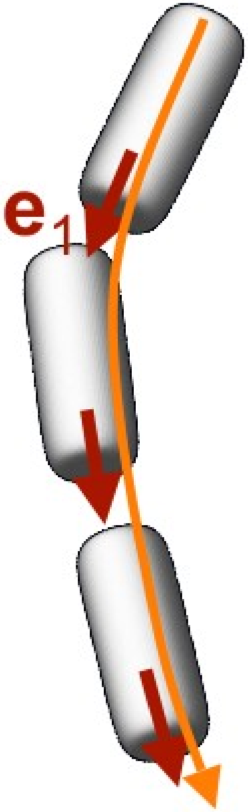
Methods:

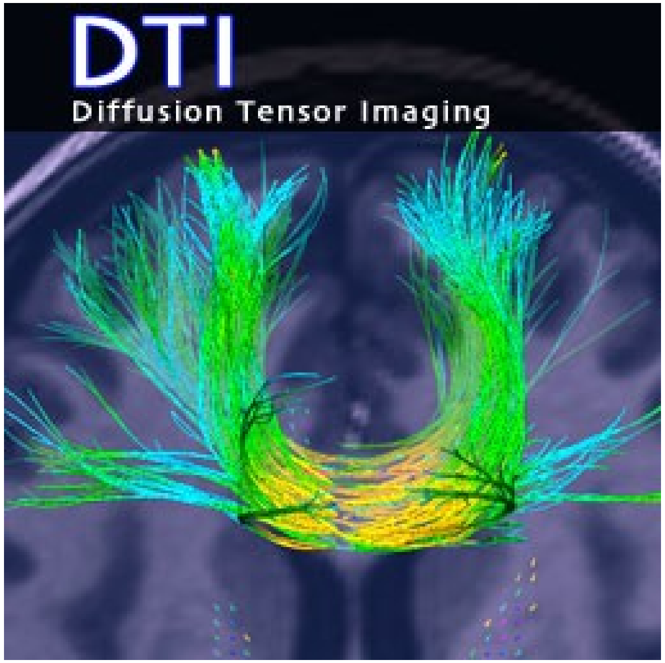
- Principal Diffusion Direction Fiber Tracking
- **Geodesic Approaches**

Fiber Tracking[1]

Path integration along principal eigenvector







Riemannian Manifold

Definition from Wikipedia:
Riemannian manifold (M,g) is
• Differentiable manifold M
• Each tangent space of M is equipped with an inner product g, a Riemannian metric,
•The metric g is a positive definite symmetric tensor: a metric tensor.

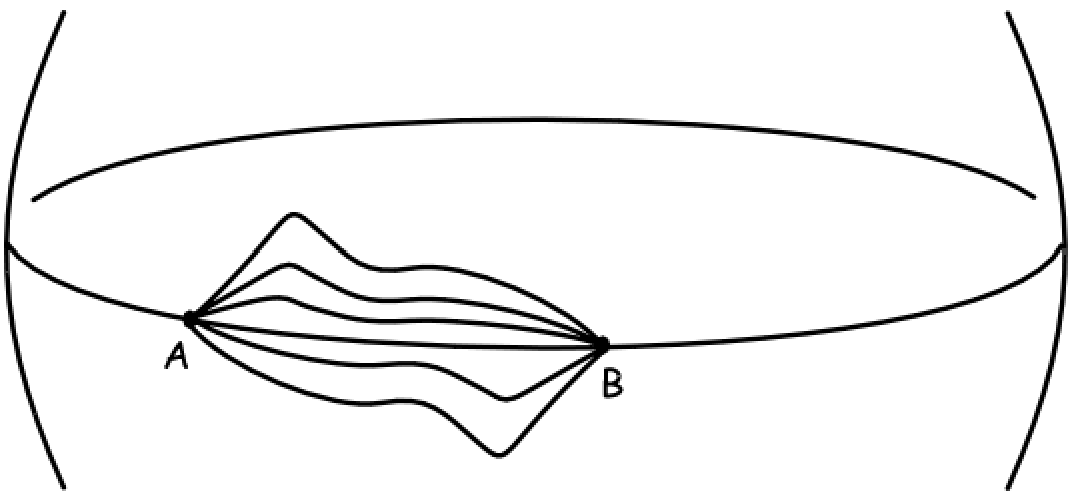
In other words, a Riemannian manifold is a differentiable manifold in which the tangent space at each point is a Euclidean space.

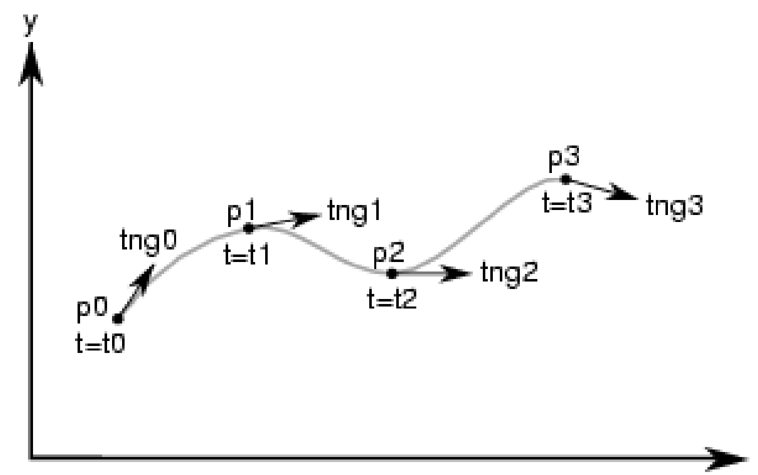
Geodesic

The geodesic between two points is computed by minimizing the Energy

$$E = \int_0^1 \langle T(t), T(t) \rangle_{g(x)} dt$$

where T is the tangent vector

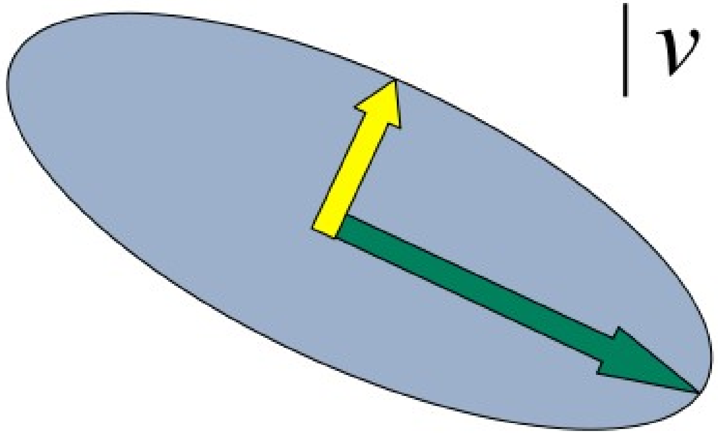


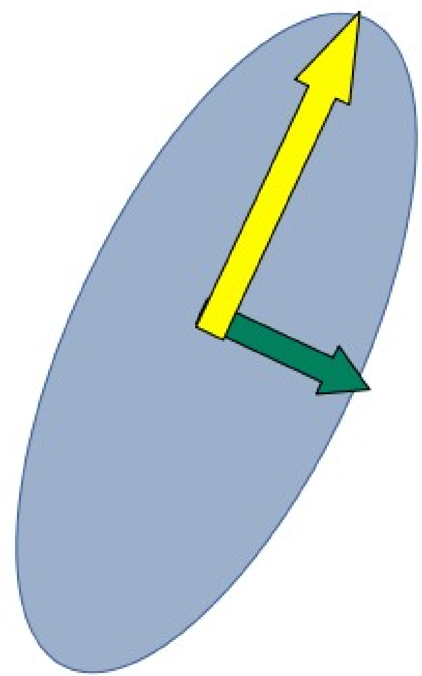


Geodesic Connectivity[1]

Connectivity should be proportional to distance in some metric space.

$$|v|^2 = v^T G v$$





Diffusion Tensor, D

Metric Tensor, $G = D^{-1}$

O'Donnell, Haker, Westin, MICCAI 2002

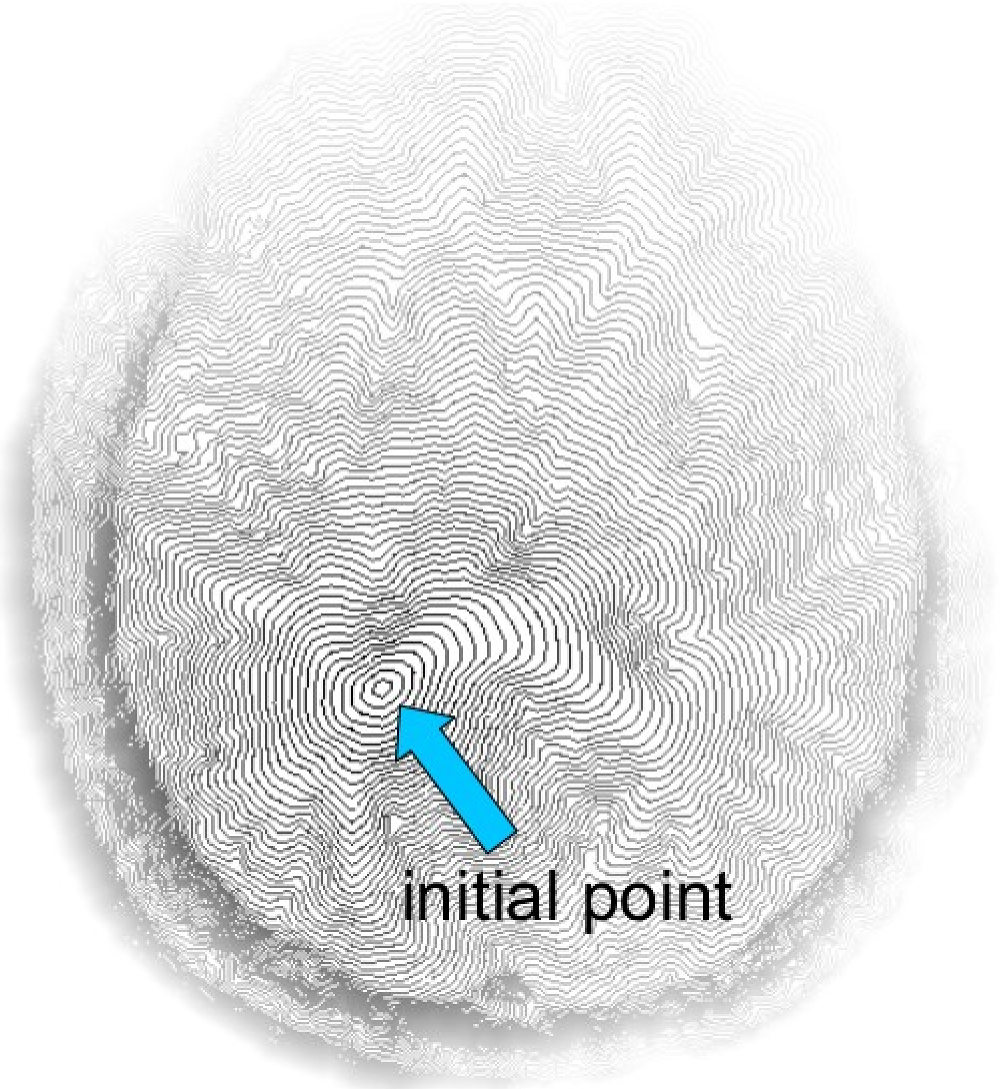
Riemannian Distance Map[1]

Input: Riemannian metric tensor G.

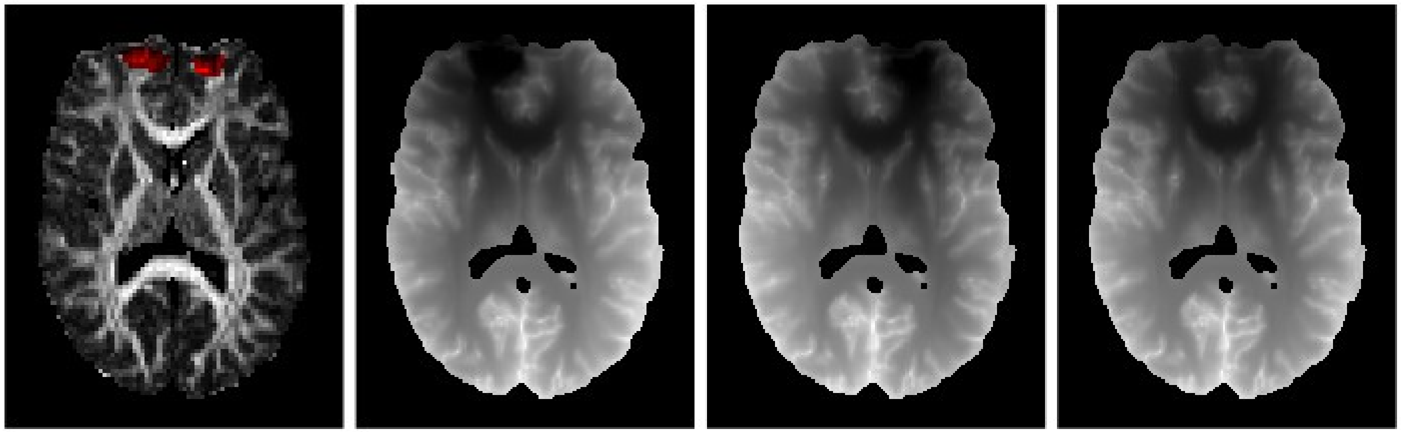
Input: initial point

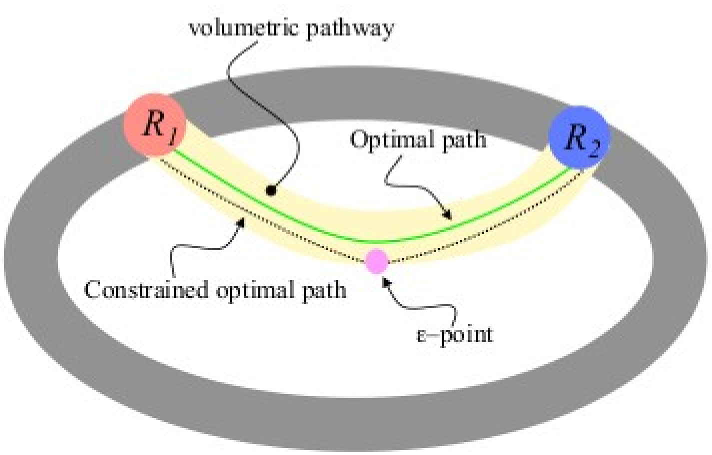
Output: geodesic paths.

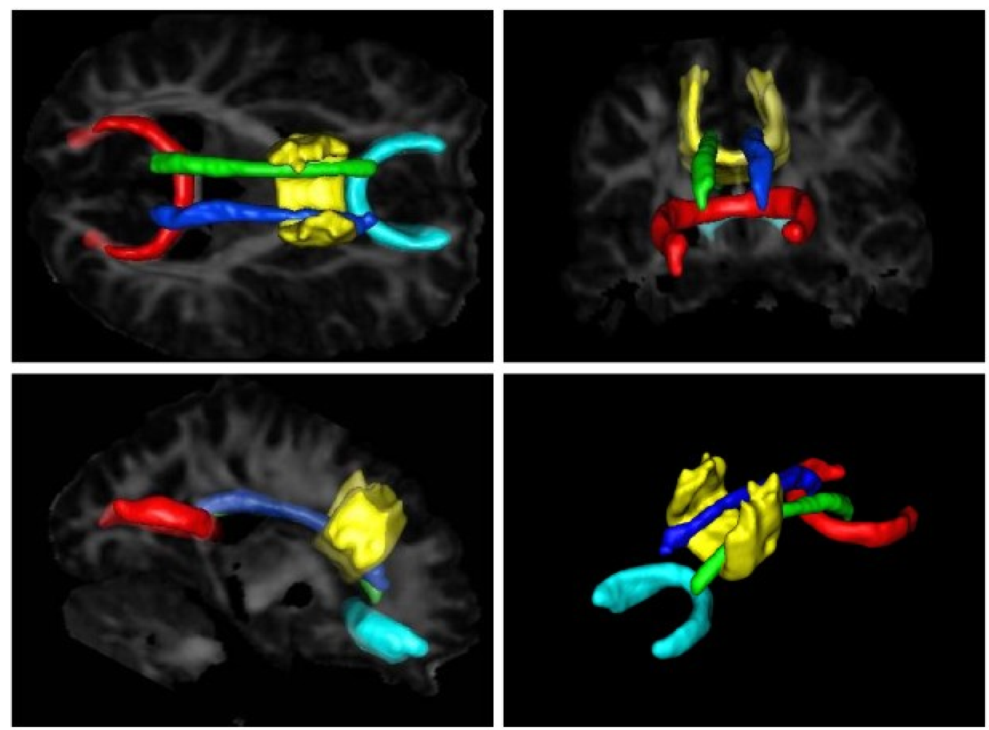
Output: distances between points in the brain.



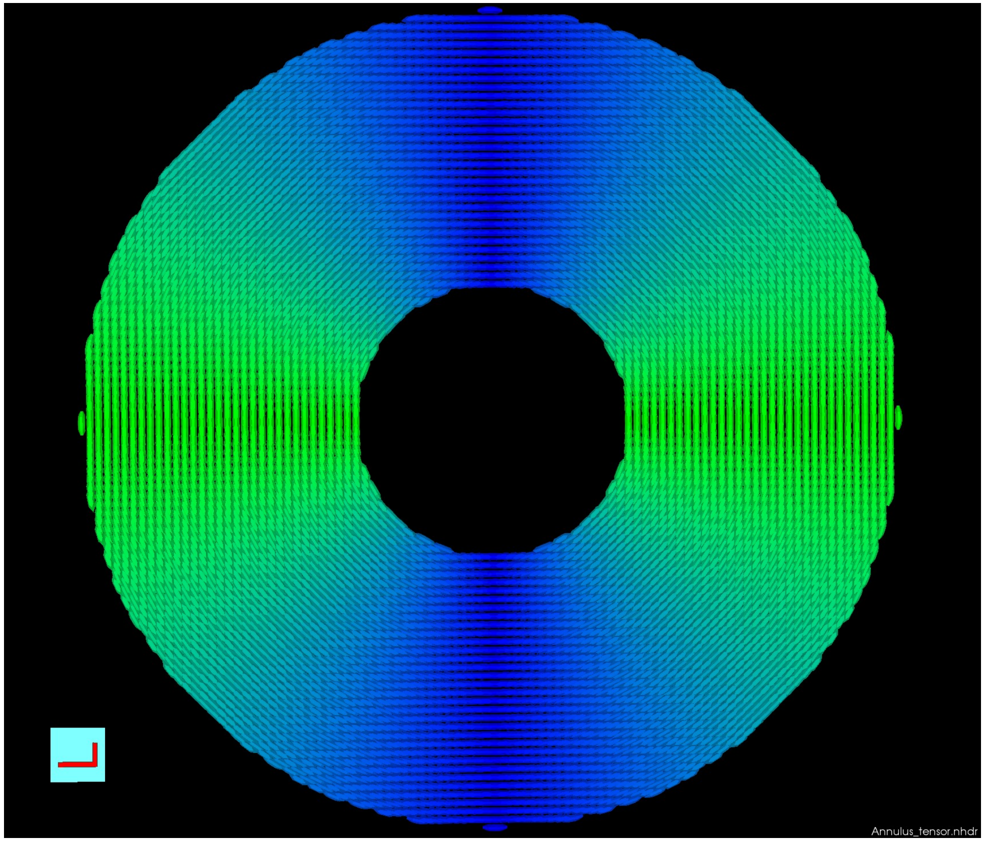
Geodesic Connectivity[2]

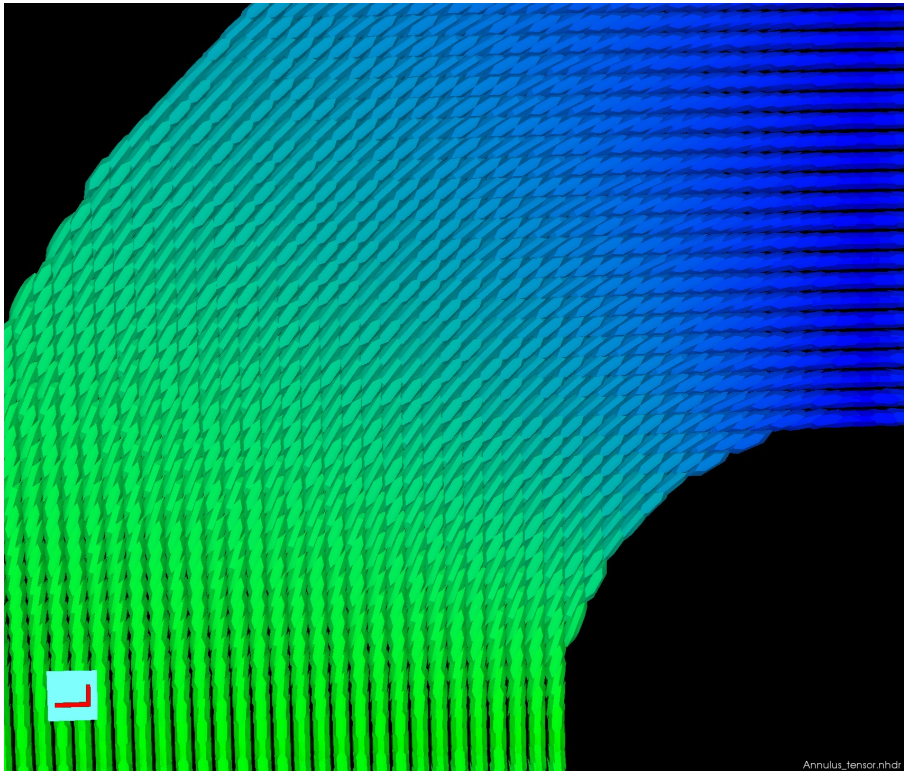




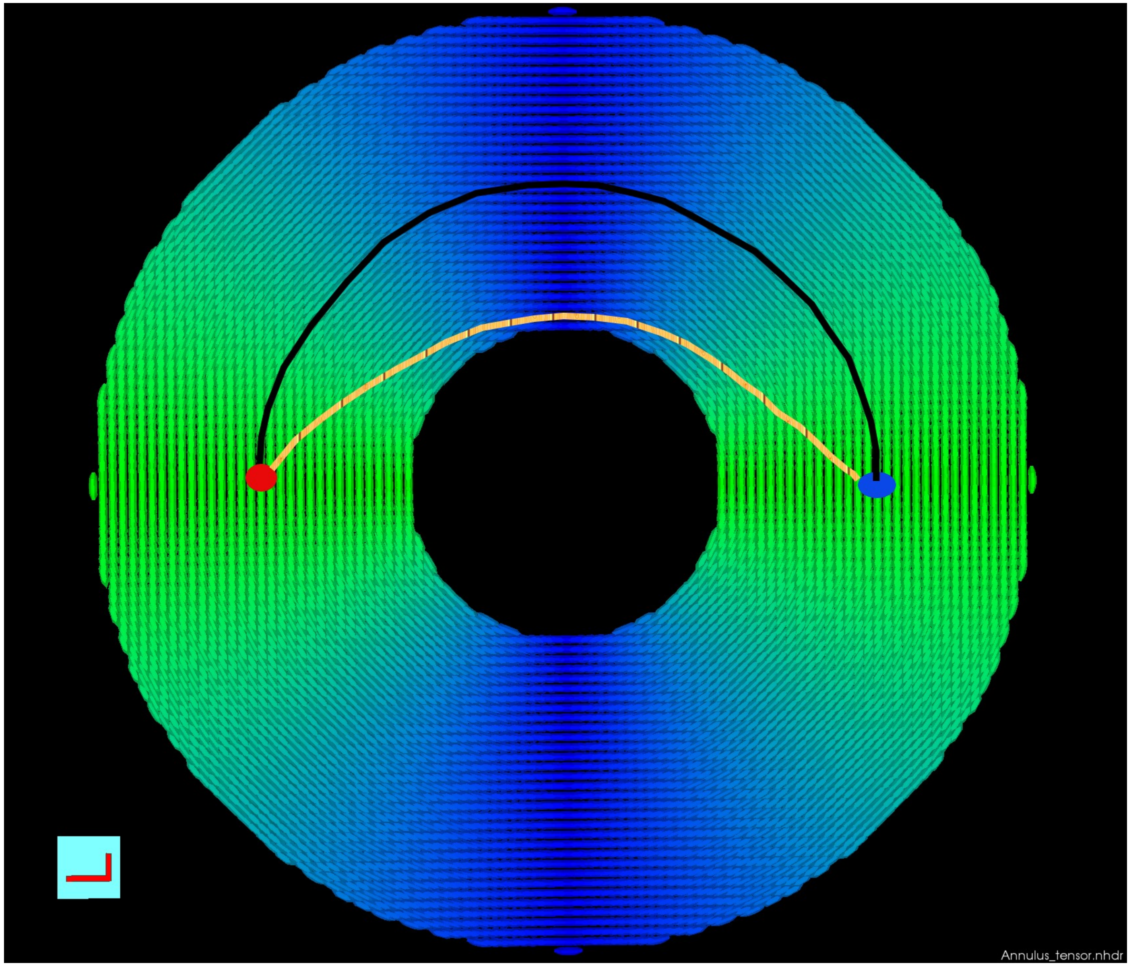


Synthetic DTI

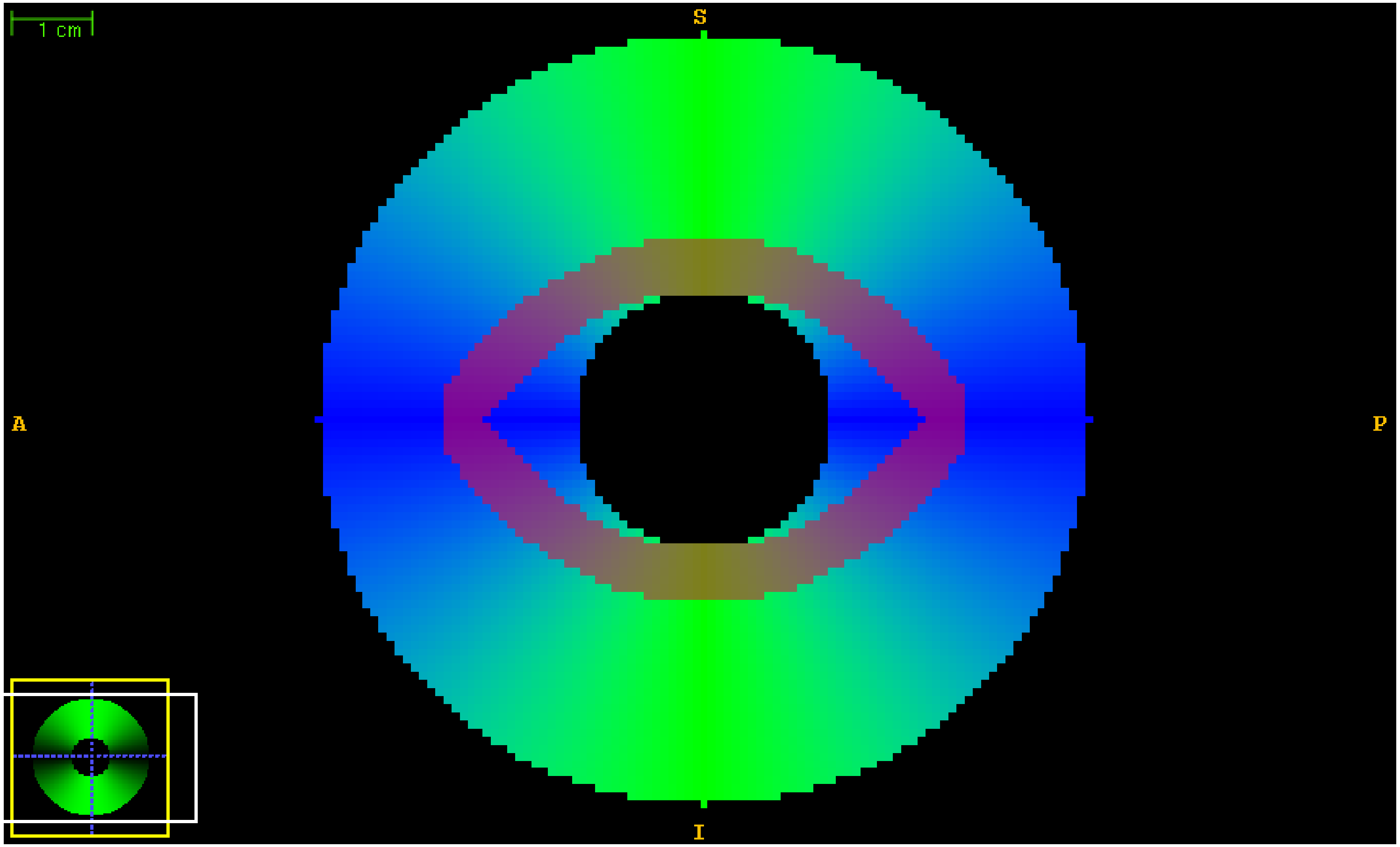


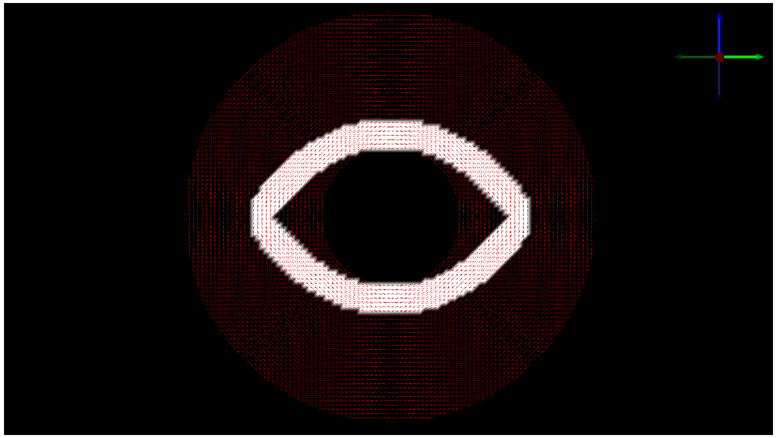


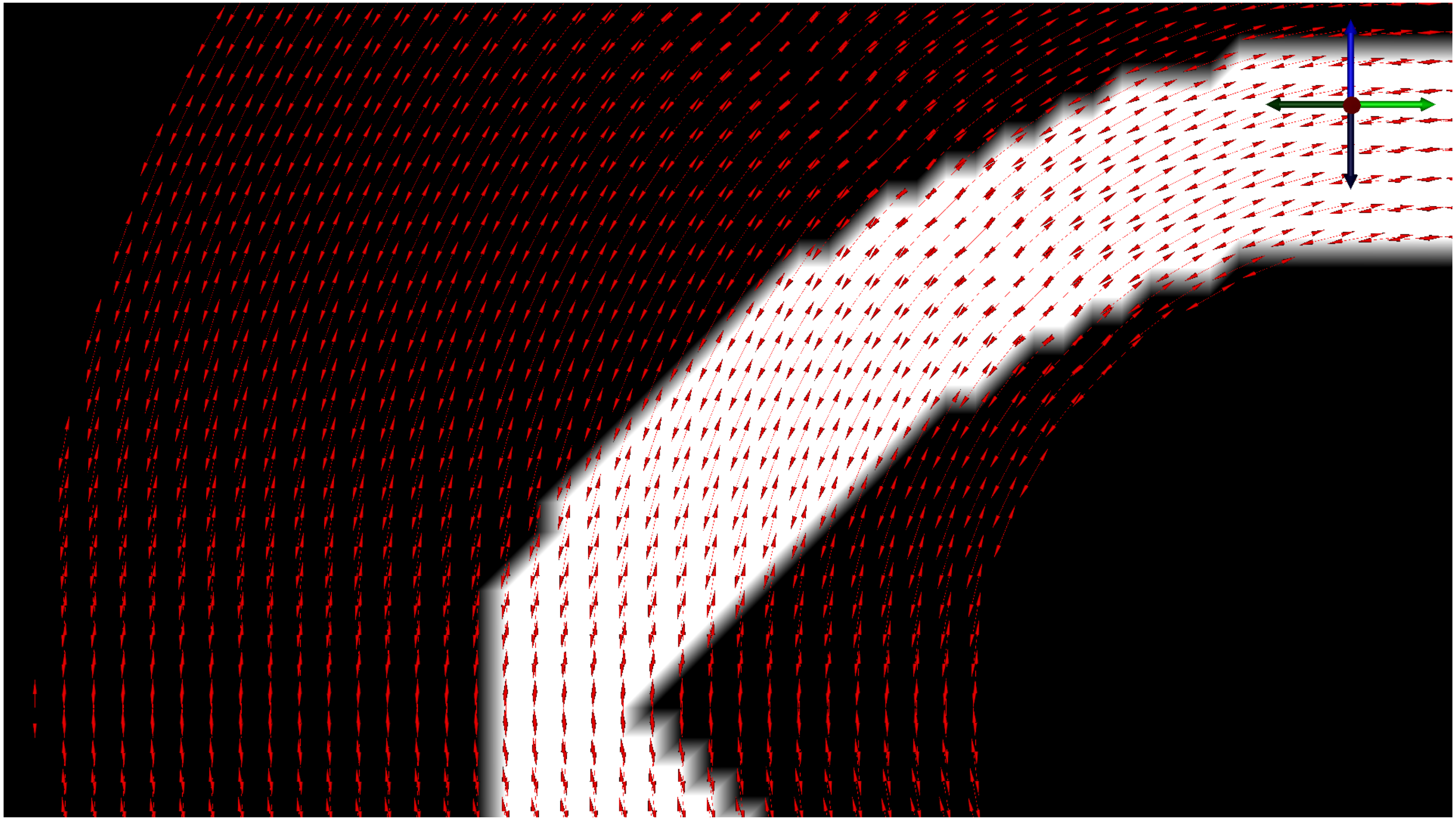
What's the problem?



The geodesic takes a shortcut







We want the geodesic to follow the principal eigenvectors

On a Riemannian manifold, the geodesic between two points is defined by the minimization of the energy functional $E = \int_0^1 \langle T(t), T(t) \rangle_{g(x)} dt$, where $g(x)$ is the Riemannian metric, and $g(x) = D(x)^{-1}$ in DT-MRI, where $D(x)$ is the symmetric, positive-definite matirx at each point $x \in \Omega$. However, in DT-MRI, the geodesic computed by minimizing the functional E does not follow the principal eigenvector at each point x. So, we want to find a scalar field $f(x)$ to scale the metric at each point, then the energy functional becomes $E = \int_0^1 \langle T(t), T(t) \rangle_{f(x)g(x)} dt = \int_0^1 f(x) \langle T(t), T(t) \rangle_{g(x)} dt$. After the scaling, we want the computed geodesic to follow the principal eigenvector at each point x. Since $f(x)$ should be positive, so use $f(x) = e^{\alpha(x)}$.

Math

The energy functional is $E(T) = \int_0^1 e^\alpha \langle T, T \rangle dt$

$$\nabla_V \int_0^1 e^\alpha \langle T, T \rangle dt$$
$$= \int_0^1 \nabla_V e^\alpha \langle T, T \rangle dt$$
$$= \int_0^1 \nabla_V e^\alpha \cdot \langle T, T \rangle + e^\alpha \nabla_V \langle T, T \rangle dt$$
$$= \int_0^1 \langle V, \operatorname{grad} e^\alpha \rangle \cdot \langle T, T \rangle + 2e^\alpha \langle \nabla_V T, T \rangle dt$$
$$= \int_0^1 \langle V, \operatorname{grad} e^\alpha \rangle \cdot ||T||^2 + 2e^\alpha \langle \nabla_V T, T \rangle dt$$
$$= \int_0^1 \langle V, \operatorname{grad} e^\alpha \cdot ||T||^2 \rangle + 2\langle \nabla_V T, e^\alpha T \rangle dt$$
$$= \int_0^1 \langle V, \operatorname{grad} e^\alpha \cdot ||T||^2 \rangle - 2\langle V, \nabla_T e^\alpha \cdot T + e^\alpha \nabla_T T \rangle dt$$
$$= \int_0^1 \langle V, \operatorname{grad} e^\alpha \cdot ||T||^2 \rangle - 2\langle V, de^\alpha(T) \cdot T + e^\alpha \nabla_T T \rangle dt$$
$$= \int_0^1 \langle V, \operatorname{grad} e^\alpha \cdot ||T||^2 - 2de^\alpha(T) \cdot T - 2e^\alpha \nabla_T T \rangle dt$$

Math Cont.

For any V, $\langle V, \operatorname{grad} e^\alpha \cdot ||T||^2 - 2de^\alpha(T) \cdot T - 2e^\alpha \nabla_T T \rangle$ should be 0,

We get

$$\operatorname{grad} e^\alpha \cdot ||T||^2 - 2de^\alpha(T) \cdot T - 2e^\alpha \nabla_T T = 0$$
$$\Rightarrow e^\alpha \nabla_T T = \frac{1}{2} \operatorname{grad} e^\alpha \cdot ||T||^2 - de^\alpha(T) \cdot T$$
$$\Rightarrow e^\alpha \nabla_T T = \frac{1}{2} e^\alpha \operatorname{grad} \alpha \cdot ||T||^2 - e^\alpha d\alpha(T) \cdot T$$

Since $e^\alpha \neq 0$, we can divide both sides by e^α and simplfy the above equation to:

$$\nabla_T T = \frac{1}{2} \operatorname{grad} \alpha \cdot ||T||^2 - d\alpha(T) \cdot T$$

Math Cont.

$\nabla_T T = \frac{1}{2}\,\text{grad}\,\alpha \cdot ||T||^2 - d\alpha(T) \cdot T$

In the case of T has unit length, actually we can normalize $||T||$, then direction of $\nabla_T T$ will be normal to T. Then we can decompose $\text{grad}\,\alpha$ into two components, $\text{grad}\,\alpha^{\perp}$ in the direction normal to T, and $\langle \text{grad}\,\alpha, T\rangle \cdot T$ in the T direction. Then, we obtain

$$\begin{cases} \text{grad}\,\alpha^{\perp} = 2\nabla_T T \\ \langle \text{grad}\,\alpha, T\rangle \cdot T = 2d\alpha(T) \cdot T = 2\langle \text{grad}\,\alpha, T\rangle \cdot T \Rightarrow \langle \text{grad}\,\alpha, T\rangle = 0 \end{cases}$$

In the end, the above equation is simplified to $\text{grad}\,\alpha = 2\nabla_T T$ But given a vector field, there may not exsit a function, whose gradients are equal to the vector field. That's why we want to minimize $||\,\text{grad}\,\alpha - 2\nabla_T T||$

The energy functional is $E(\alpha) = \int_\Omega ||\,\text{grad}\,\alpha - 2\nabla_T T||^2 dx$

$$\begin{aligned}
&\tfrac{d}{d\epsilon} \int_\Omega ||\,\text{grad}(\alpha + \epsilon h) - 2\nabla_T T||^2 dx|_{\epsilon=0} \\
&= 2\int_\Omega \langle \tfrac{d}{d\epsilon}\,\text{grad}(\alpha + \epsilon h), \text{grad}(\alpha + \epsilon h) - 2\nabla_T T\rangle dx|_{\epsilon=0} \\
&= 2\int_\Omega \langle \text{grad}\,h, \text{grad}\,\alpha - 2\nabla_T T\rangle dx \\
&= -2\int_\Omega \langle h, \text{div}(\text{grad}\,\alpha - 2\,\text{div}(\nabla_T T)\rangle dx
\end{aligned}$$

So, $\text{div}(\text{grad}\,\alpha) = 2\,\text{div}(\nabla_T T)$

Boundary Condition

The boundary condition for the PDE is

$$\frac{\partial \alpha}{\partial \overrightarrow{n}} = \langle \operatorname{grad} \alpha, \overrightarrow{n} \rangle_g = \langle 2\nabla_T T, \overrightarrow{n} \rangle_g$$

Solve the PDE Numerically

$$\begin{cases} \operatorname{div}(\operatorname{grad}\alpha) = 2\operatorname{div}(\nabla_T T) \\ \frac{\mathrm{d}\,\alpha}{\mathrm{d}\,\overrightarrow{n}} = \langle 2\nabla_T T, \overrightarrow{n}\rangle_g \end{cases}$$

The Finite Difference Method for Laplace Euqation[3]

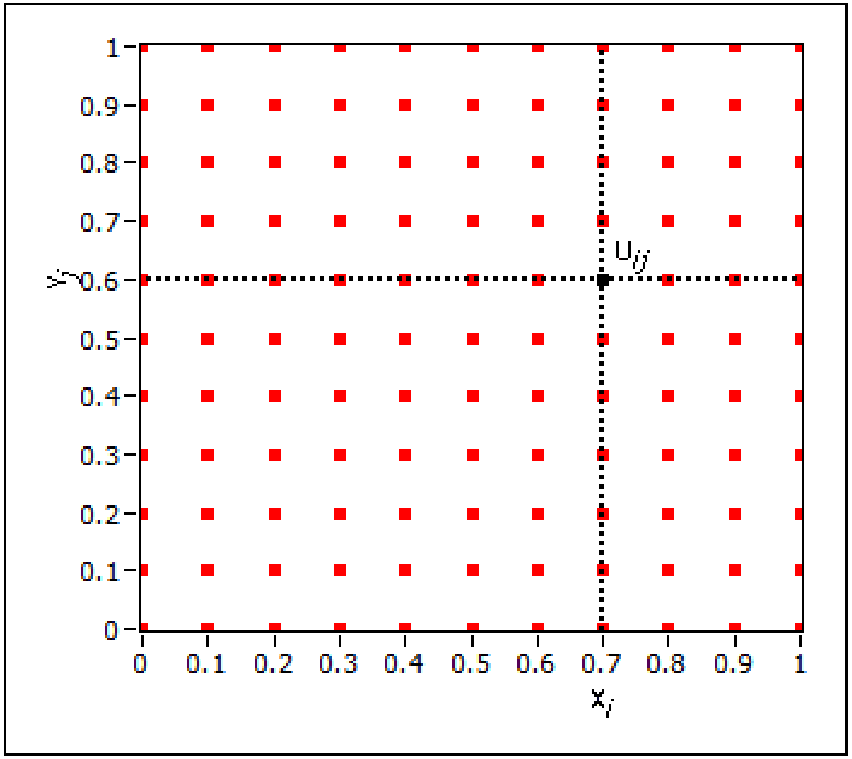
$$\begin{cases} \dfrac{\partial^2 u}{\partial^2 x} + \dfrac{\partial^2 u}{\partial^2 y} = 0, \quad 0 \le x, y \le 1 \\ u(0, y) = -y^2, \quad u(1, y) = 1 - y^2 \\ u(x, 0) = x^2, \quad u(x, 1) = x^2 - 1 \end{cases}$$

Step 1

Step One

Separate the square domain with the uniform mesh grid of $\{x_0, x_1, \ldots, x_n\} \times \{y_0, y_1, \ldots, y_n\}$ where x_0 and y_0 are 0 and x_n and y_n are 1. You can evaluate the value of the unknown function on discrete boundary points with the Dirichlet condition.

The following illustration shows the mesh grid when n is 10.



Step 2

Step Two

Approximate the Laplace equation by the second order central difference scheme. The Laplace equation becomes

$$\frac{1}{h^2}\left(u_{i+1,j} - 2u_{ij} + u_{i-1,j}\right) + \frac{1}{h^2}\left(u_{i,j+1} - 2u_{ij} + u_{i,j-1}\right) = 0 \quad \textbf{(B)}$$

or

$$u_{i-1,j} + u_{i,j-1} - 4u_{ij} + u_{i,j+1} + u_{i+1,j} = 0 \quad \textbf{(C)}$$

where u_{ij} denotes the value of u on point (x_i, y_j). The second formula is also known as the five-point formula because it is a linear combination of the values of u evaluated on five points.

Step 3

Step Three

Formulate the Laplace equation by combining all difference equations of u_{ij}. Because the boundary condition specifies the values of u on the boundary, you can move the values to the right side of the equation, which generates this side of the equation. If the condition is Neumann, approximate the value of u on the boundary with its normal derivative.

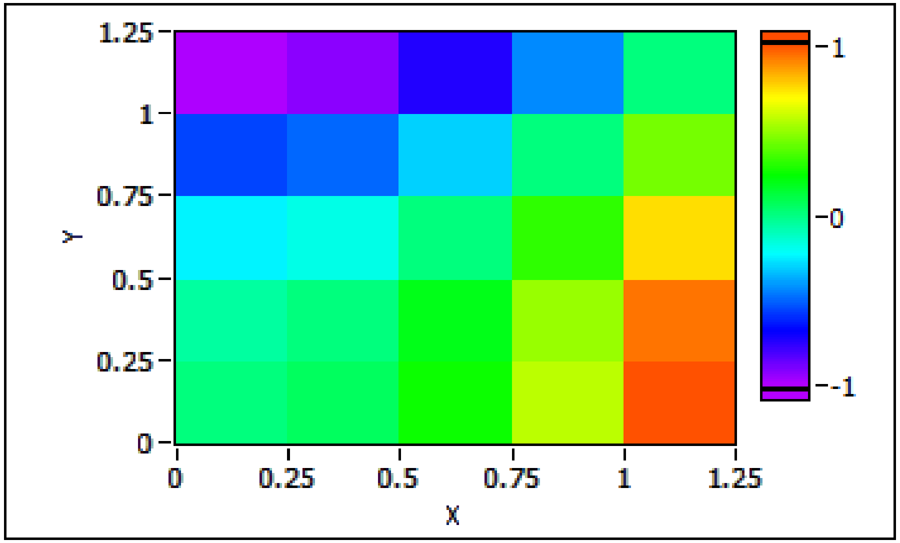
When n is 4, you have the following 9-by-9 linear equation:

$$\begin{bmatrix} 4 & -1 & & -1 & & & & & \\ -1 & 4 & -1 & & -1 & & & & \\ & -1 & 4 & & & -1 & & & \\ -1 & & & 4 & -1 & & -1 & & \\ & -1 & & -1 & 4 & -1 & & -1 & \\ & & -1 & & -1 & 4 & & & -1 \\ & & & -1 & & & 4 & -1 & \\ & & & & -1 & & -1 & 4 & -1 \\ & & & & & -1 & & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.25 \\ 1.5 \\ -0.25 \\ 0 \\ 0.75 \\ -1.5 \\ -0.75 \\ 0 \end{bmatrix} \quad \textbf{(D)}$$

Step 4

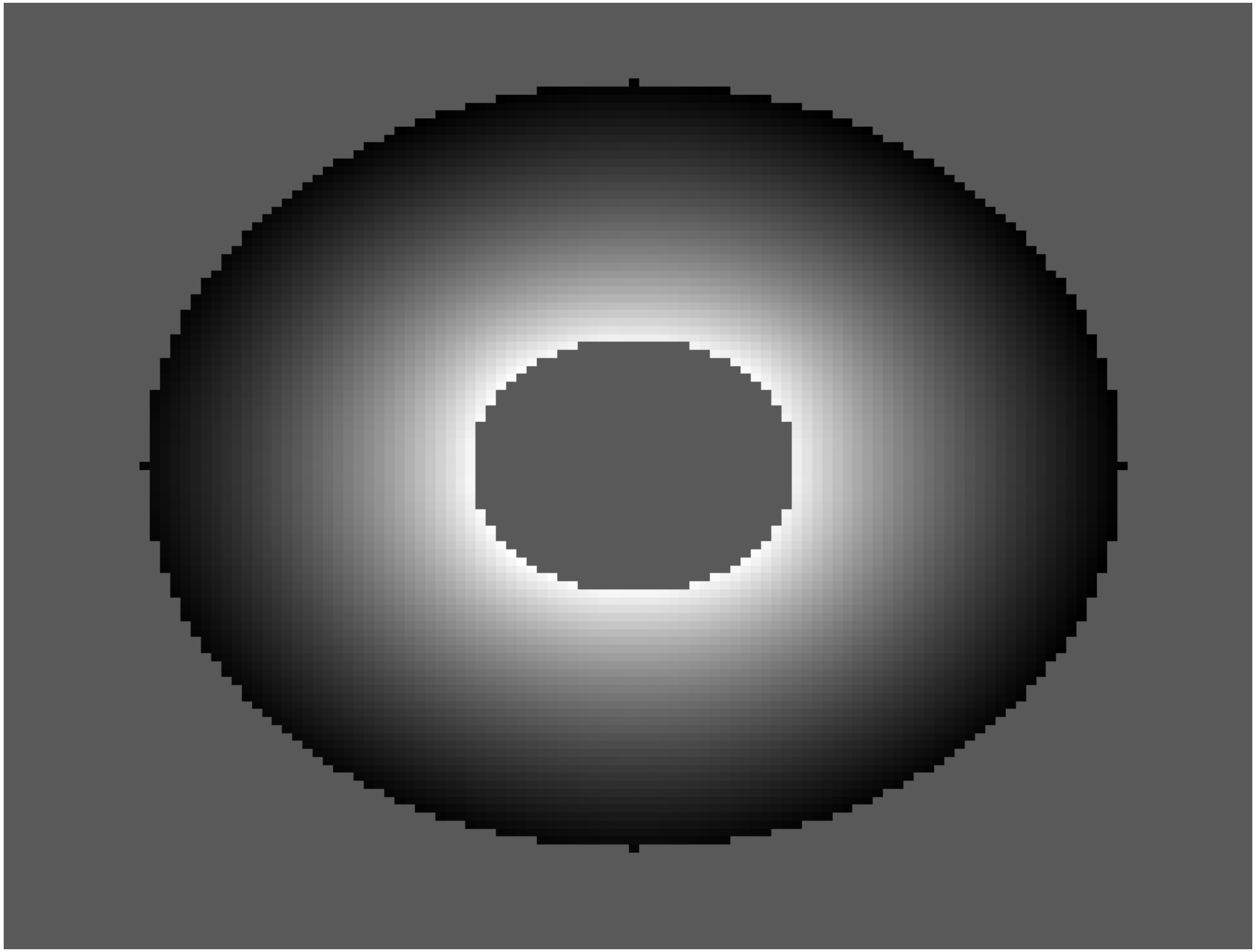
Step Four

Solve the 9-by-9 linear equation to get the approximate solution of the Laplace equation on a mesh grid, shown as follows.

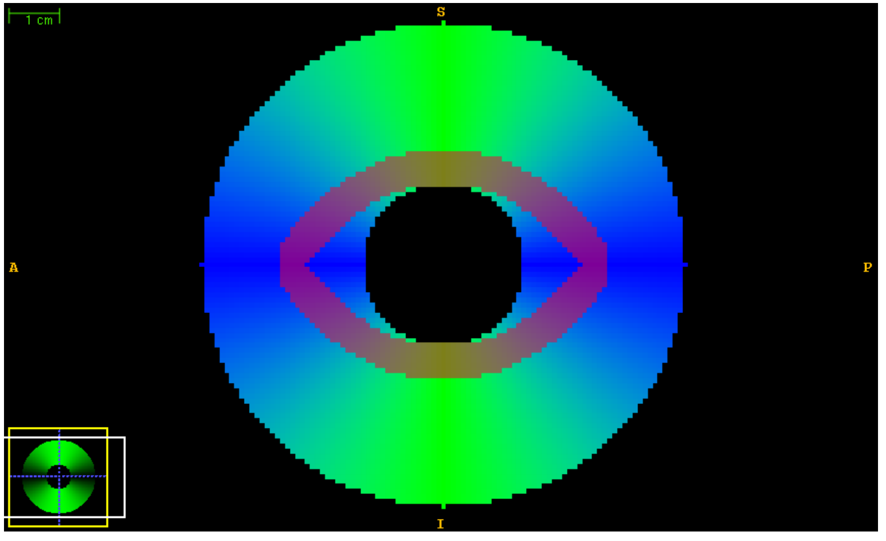


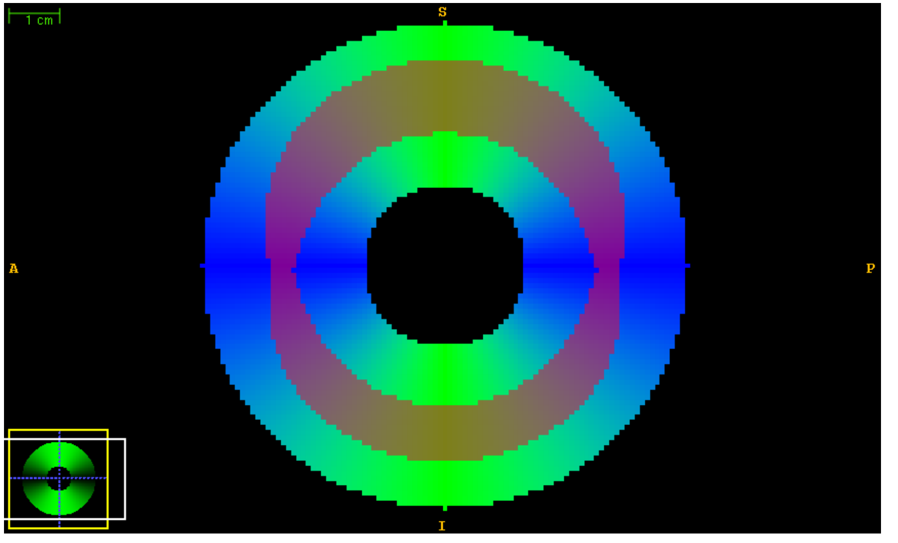
The coefficient matrix of the linear equation, which you deduce on a uniform mesh grid from the PDE, has a special structure, such as the tri-diagonal or banded tri-diagonal structures. A compact storage scheme always stores the coefficient matrix, meaning you can use fast solvers for the linear equation with the structured coefficient matrix.

Results: Alpha field



Results: Segmentation





References

1. MICCAI Diffusion MRI Tutorial, MICCAI 2007
2. A Volumetric Approach to Quantifying Region-to-Region White Matter Connectivity in Diffusion Tensor MRI, IPMI 2007
3. Solving PDEs with Numerical Methods, http://zone.ni.com/reference/en-XX/help/371361G-01/lvanlsconcepts/methods_solve_pdes/